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CS101 Calculus 2 Section C - Homework 10

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Exercise 1 (100 points).

Study the convergence of the following series:

a) $\sum_{n=1}^{\infty} \frac{n!}{1992^n}$;

$$\lim_{n\to\infty} \frac{n!}{1992^n} = \lim_{n\to\infty} \frac{1\cdot 2\cdot \ldots \cdot n}{1992\cdot 1992\cdot \ldots \cdot 1992} \longrightarrow \infty \overset{\text{term test}}{\Longrightarrow} \sum_{n=1}^{\infty} \frac{n!}{1992^n} \longrightarrow \text{diverges}$$

b) $\sum_{n=1}^{\infty} \frac{n^2-1}{\sqrt{n^4+2n^2+5}}$;

$$\lim_{n\to\infty} \frac{n^2-1}{\sqrt{n^4+2n^2+5}} = \lim_{n\to\infty} \frac{n^2-1}{n^2\sqrt{1+\frac{2}{n^2}+\frac{5}{n^4}}} \longrightarrow 1 \overset{\text{term test}}{\Longrightarrow} \sum_{n=1}^{\infty} \frac{n^2-1}{\sqrt{n^4+2n^2+5}} \longrightarrow \text{diverges}$$

c) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{\sqrt{n+2}}$;

$$\frac{\sqrt{n+1}-\sqrt{n-1}}{\sqrt{n+2}} = \frac{(\sqrt{n+1}-\sqrt{n-1})}{\sqrt{n+2}} \cdot \frac{(\sqrt{n+1}+\sqrt{n-1})}{(\sqrt{n+1}+\sqrt{n-1})} = \frac{2}{\sqrt{n+2}(\sqrt{n+1}+\sqrt{n-1})}$$

$$\lim_{n\to\infty} \frac{\frac{2}{\sqrt{n+2}(\sqrt{n+1}+\sqrt{n-1})}}{\frac{1}{n}} = \frac{2n}{n\sqrt{1+\frac{2}{n}}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}\right)} = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \overset{1\leq 1}{\longrightarrow} \text{diverges} \overset{\text{LCT}}{\Longrightarrow} \sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{\sqrt{n+2}} \longrightarrow \text{diverges}$$

d) $\sum_{n=2}^{\infty} \frac{\sin n+2}{\sqrt[4]{2n^5-2}}$;

$$\frac{\sin n+2}{\sqrt[4]{2n^5-2}} \leq \frac{3}{\sqrt[4]{2n^5-2}}, \text{ and } \lim_{n\to\infty} \frac{\frac{3}{\sqrt[4]{2n^5-2}}}{\frac{3}{\sqrt[4]{2}n^{\frac{5}{4}}}} = 1$$

$$\sum_{n=1}^{\infty} \frac{3}{\sqrt[4]{2}} \frac{1}{n^{\frac{5}{4}}} \overset{\frac{5}{4}>1}{\longrightarrow} \text{converges} \overset{\text{LCT}}{\Longrightarrow} \sum_{n=2}^{\infty} \frac{3}{\sqrt[4]{2n^5-2}} \longrightarrow \text{converges} \overset{\text{DCT}}{\Longrightarrow} \sum_{n=2}^{\infty} \frac{\sin n+2}{\sqrt[4]{2n^5-2}} \longrightarrow \text{converges}$$

e) $\sum_{n=1}^{\infty}\left(\frac{2n^3-1}{2n^3-n}\right)^{-n}$;

$$\lim_{n\to\infty}\left(\frac{2n^3-1}{2n^3-n}\right)^{-n}=\lim_{n\to\infty}\left(1-\frac{n-1}{2n^3-n}\right)^{-n\cdot\frac{n-1}{2n^3-n}\cdot\frac{2n^3-n}{n-1}}=\lim_{n\to\infty}e^{\frac{n^2-n}{2n^3-n}}=1$$

$$\therefore \text{ by term test } \sum_{n=1}^{\infty}\left(\frac{2n^3-1}{2n^3-n}\right)^{-n}\longrightarrow \text{diverges}$$

f) $\sum_{n=1}^{\infty}\cos\frac{2021}{n+2022}$;

$$\lim_{n\to\infty}\cos\frac{2021}{n+2022}=\cos 0=1\overset{\text{term test}}{\Longrightarrow}\sum_{n=1}^{\infty}\cos\frac{2021}{n+2022}\longrightarrow \text{diverges}$$

g) $\sum_{n=1}^{\infty}\frac{5+\sin\frac{1}{n}}{2^n-n}$;

$$\frac{5+\sin\frac{1}{n}}{2^n-n}\leq\frac{6}{2^n-n}\text{ and }\lim_{n\to\infty}\frac{\frac{6}{2^n-n}}{\frac{6}{2^n}}=\frac{2^n}{2^n-n}=\frac{1}{1-\frac{n}{2^n}}=1$$

$$\sum_{n=1}^{\infty}6\left(\frac{1}{2}\right)^n\overset{|\frac{1}{2}|<1}{\longrightarrow}\text{converges}\overset{\text{LCT}}{\Longrightarrow}\sum_{n=1}^{\infty}\frac{6}{2^n-n}\longrightarrow\text{converges}\overset{\text{DCT}}{\Longrightarrow}\sum_{n=1}^{\infty}\frac{5+\sin\frac{1}{n}}{2^n-n}\longrightarrow\text{converges}$$

h) $\sum_{n=1}^{\infty}\frac{5^n+4n^5}{6^n+n^6}$;

$$\lim_{n\to\infty}\sqrt[n]{\frac{5^n+4n^5}{6^n+n^6}}=\lim_{n\to\infty}\frac{5\sqrt[n]{1+\frac{4n^5}{5^n}}}{6\sqrt[n]{1+\frac{n^6}{6^n}}}=\frac{5}{6}\overset{\text{root test}}{\Longrightarrow}\sum_{n=1}^{\infty}\frac{5^n+4n^5}{6^n+n^6}\longrightarrow\text{converges}$$

i) $\sum_{n=1}^{\infty}\left(\frac{5n+3}{2+5n}\right)^{n^2}$;

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{5n+3}{2+5n}\right)^{n^2}}=\lim_{n\to\infty}\left(\frac{5n+2+1}{2+5n}\right)^{n}=\lim_{n\to\infty}\left(1+\frac{1}{2+5n}\right)^{n\cdot\frac{2+5n}{2+5n}}$$

$$\lim_{n\to\infty}e^{\frac{n}{2+5n}}=e^{\frac{1}{5}}>1\overset{\text{root test}}{\Longrightarrow}\sum_{n=1}^{\infty}\left(\frac{5n+3}{2+5n}\right)^{n^2}\longrightarrow\text{diverges}$$

j) $\sum_{n=1}^{\infty}\left(\frac{2n^3-1}{2n^3+2}\right)^{n^4}$;

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{2n^3-1}{2n^3+2}\right)^{n^4}}=\lim_{n\to\infty}\left(1-\frac{3}{2n^3+2}\right)^{-\frac{2n^3+2}{3}\cdot n^3\cdot-\frac{3}{2n^3+2}}=\lim_{n\to\infty}e^{-\frac{3n^3}{2n^3+2}}=\frac{1}{e^{\frac{3}{2}}}<1$$

$$\therefore \text{ by root test } \sum_{n=1}^{\infty}\left(\frac{2n^3-1}{2n^3+2}\right)^{n^4}\longrightarrow\text{converges}$$

k) $\sum_{n=1}^{\infty} \frac{1}{n^{10}} \left(\frac{6}{5}\right)^n$;

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n^{10}}\left(\frac{6}{5}\right)^n} = \lim_{n\to\infty} \frac{6}{5}\left(\frac{1}{\sqrt[n]{n}}\right)^{10} = \frac{6}{5} \overset{\text{root test}}{\Longrightarrow} \sum_{n=1}^{\infty} \frac{1}{n^{10}}\left(\frac{6}{5}\right)^n \longrightarrow \text{diverges}$$

l) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln^2 n+10}$;

$$\lim_{n\to\infty} \frac{1}{\ln^2 n + 10} = 0 \text{ and } \ln n \uparrow \Longrightarrow \ln^2 n + 10 \uparrow \Longrightarrow \frac{1}{\ln^2 n + 10} \downarrow$$

$$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{\ln^2 n + 10} \longrightarrow \text{converges}$$

m) $\sum_{n=1}^{\infty} \frac{(2n+1)!!}{(3)^n n!}$;

$$a_n = \frac{(2n+1)!!}{(3)^n n!}\ a_{n+1} = \frac{(2(n+1)+1)!!}{(3)^(n+1)(n+1)!}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{\frac{\cancel{1}\cdot\cancel{3}\cdot\ldots\cdot\cancel{(2n+1)}\cdot(2n+3)}{3\cancel{(3)^n}\cdot(\cancel{1}\cdot\cancel{2}\cdot\ldots\cdot\cancel{n}\cdot(n+1))}}{\frac{\cancel{1}\cdot\cancel{3}\cdot\ldots\cdot\cancel{(2n+1)}}{\cancel{(3)^n}\cdot(\cancel{1}\cdot\cancel{2}\cdot\ldots\cdot\cancel{n})}} = \lim_{n\to\infty} \frac{2n+3}{3(n+1)} = \frac{2}{3} \overset{\text{ratio test}}{\Longrightarrow} \sum_{n=1}^{\infty} \frac{(2n+1)!!}{(3)^n n!} \longrightarrow \text{converges}$$

n) $\sum_{n=1}^{\infty} \frac{\sin \pi n}{\sqrt[3]{4n+1}}$;

$$\sin \pi n = 0, \forall n \in \mathbb{N} \Longrightarrow \sum_{n=1}^{\infty} \frac{\sin \pi n}{\sqrt[3]{4n+1}} = 0$$

o) $\sum_{n=5}^{\infty} \frac{1\cdot 4\cdot 7\cdot\ldots\cdot(3n-2)}{2\cdot 6\cdot 10\cdot\ldots\cdot(4n-2)}$;

$$\lim_{n\to\infty} \frac{\frac{\cancel{1}\cdot\cancel{4}\cdot\cancel{7}\cdot\ldots\cdot\cancel{(3n-2)}\cdot(3n+1)}{\cancel{2}\cdot\cancel{6}\cdot\cancel{10}\cdot\ldots\cdot\cancel{(4n-2)}\cdot(4n+2)}}{\frac{\cancel{1}\cdot\cancel{4}\cdot\cancel{7}\cdot\ldots\cdot\cancel{(3n-2)}}{\cancel{2}\cdot\cancel{6}\cdot\cancel{10}\cdot\ldots\cdot\cancel{(4n-2)}}} = \lim_{n\to\infty} \frac{3n+1}{4n+2} = \frac{3}{4} \overset{\text{ratio test}}{\Longrightarrow} \sum_{n=5}^{\infty} \frac{1\cdot 4\cdot 7\cdot\ldots\cdot(3n-2)}{2\cdot 6\cdot 10\cdot\ldots\cdot(4n-2)} \longrightarrow \text{converges}$$

p) $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{4n-2}} - \frac{1}{\sqrt{4n+2}}\right)$;

$$\frac{1}{\sqrt{4n-2}} - \frac{1}{\sqrt{4n+2}} = \frac{\sqrt{4n+2}-\sqrt{4n-2}}{\sqrt{4n-2}\sqrt{4n+2}} = \frac{4}{\sqrt{4n-2}\sqrt{4n+2}(\sqrt{4n+2}+\sqrt{4n-2})}$$

$$\lim_{n\to\infty} \frac{\frac{4}{\sqrt{4n-2}\sqrt{4n+2}(\sqrt{4n+2}+\sqrt{4n-2})}}{\frac{1}{4n^{\frac{3}{2}}}} = \frac{16n^{\frac{3}{2}}}{\sqrt{4n-2}\sqrt{4n+2}(\sqrt{4n+2}+\sqrt{4n-2})} = 1$$

$$\sum_{1}^{\infty} \frac{1}{16n^{\frac{3}{2}}} \xrightarrow{\frac{3}{2}>1} \text{converges} \overset{\text{LCT}}{\Longrightarrow} \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{4n-2}} - \frac{1}{\sqrt{4n+2}}\right) \longrightarrow \text{converges}$$

q) $\sum_{n=2}^{\infty} \frac{\cos\frac{2}{\sqrt{n}} \sin\frac{3}{n}}{\sqrt{2n}}$;

$$\lim_{n\to\infty} \frac{\frac{\cos\frac{2}{\sqrt{n}}\sin\frac{3}{n}}{\sqrt{2n}}}{\frac{3}{\sqrt{2}n^{\frac{3}{2}}}} = \lim_{n\to\infty} \frac{\cos\frac{2}{\sqrt{n}}\cdot\sin\frac{3}{n}}{\frac{3}{n}} = 1$$

$$\sum_{n=2}^{\infty} \frac{3}{\sqrt{2}n^{\frac{3}{2}}} \xrightarrow{\frac{3}{2}>1} \text{converges} \overset{\text{LCT}}{\Longrightarrow} \sum_{n=2}^{\infty} \frac{\cos\frac{2}{\sqrt{n}}\sin\frac{3}{n}}{\sqrt{2n}} \longrightarrow \text{converges}$$

r) $\sum_{n=1}^{\infty} \sin^4 \frac{\sqrt{2n^2-1}}{n^2+1}$;

$$\lim_{n\to\infty} \frac{\sin^4 \frac{\sqrt{2n^2-1}}{n^2+1}}{\frac{4}{n^4}} = \lim_{n\to\infty} \left(\frac{\sin \frac{n\sqrt{2-\frac{1}{n^2}}}{n^2\left(1+\frac{1}{n^2}\right)}}{\frac{\sqrt{2}}{n}} \right)^4 = 1$$

$$\sum_{n=1}^{\infty} \frac{4}{n^4} \xrightarrow{4>1} \text{converges} \overset{\text{LCT}}{\Longrightarrow} \sum_{n=1}^{\infty} \sin^4 \frac{\sqrt{2n^2-1}}{n^2+1} \longrightarrow \text{converges}$$

s) $\sum_{n=5}^{\infty} \frac{1}{n \ln n (\ln \ln n)^3}$;

$$n \geq 5 \implies \frac{1}{n \ln n (\ln \ln n)^3} \geq 0 \text{ and } n \uparrow, \ln n \uparrow \implies \frac{1}{n \ln n (\ln \ln n)^3} \downarrow$$

$$\int_5^{\infty} \frac{1}{x \ln x (\ln \ln x)^3}\ \mathrm{d}x \overset{u=\ln x}{=} \int_{\ln(5)}^{\infty} \frac{1}{u(\ln u)^3}\ \mathrm{d}u \overset{v=\ln u}{=} \int_{\ln(\ln(5))}^{\infty} \frac{1}{v^3}\ \mathrm{d}v$$

$$\int_1^{\infty} \frac{1}{v^3}\ \mathrm{d}v \xrightarrow{3>1} \text{converges} \overset{\text{integral test}}{\Longrightarrow} \sum_{n=5}^{\infty} \frac{1}{n \ln n (\ln \ln n)^3} \longrightarrow \text{converges}$$

t) $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^{1.5}}$.

$$n \geq 3 \implies \frac{1}{n(\ln n)^{1.5}} \geq 0 \text{ and } n \uparrow, \ln n \uparrow \implies \frac{1}{n(\ln n)^{1.5}} \downarrow$$

$$\int_3^{\infty} \frac{1}{x(\ln x)^{1.5}}\ \mathrm{d}x \overset{u=\ln x}{=} \int_{\ln 3}^{\infty} \frac{1}{u^{\frac{3}{2}}}\ \mathrm{d}x \xrightarrow{\frac{3}{2}>1} \text{converges} \overset{\text{integral test}}{\Longrightarrow} \sum_{n=3}^{\infty} \frac{1}{n(\ln n)^{1.5}} \longrightarrow \text{converges}$$