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CS101 Calculus 2 Section C - Homework 11

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Exercise 1 (20 points).

Study the convergence of the following series:

a) $\displaystyle\sum_{n=3}^{\infty} \frac{\sin(8n-2)}{\sqrt{\ln n}}$;

$$\left|\sum_{n=1}^{N} \sin(8n-2)\right| = |\sin(6) + \sin(14) + \sin(22) + ... + \sin(8N-2)|, N \in \mathbb{N}$$

$$= \left|\frac{\cos(4)}{\cos(4)}\left(\sin(6) + \sin(14) + \sin(22) + ... + \sin(8N-2)\right)\right|$$

$$= \left|\frac{1}{2\cos(4)}\left(\cancel{\sin(10)} - \sin(4) + \cancel{\sin(18)} - \cancel{\sin(10)} + \cancel{\sin(26)} - \cancel{\sin(18)} + ... + \sin(8N+2) - \cancel{\sin(8N-6)}\right)\right|$$

$$= \left|\frac{\sin(8N+2) - \sin(4)}{2\cos 4}\right| \implies \text{ is bounded}$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{\ln n}} = 0, \ln n \uparrow \implies \sqrt{\ln n} \uparrow \implies \frac{1}{\sqrt{\ln n}} \downarrow$$

$$\therefore \sum_{n=3}^{\infty} \frac{\sin(8n-2)}{\sqrt{\ln n}} \overset{\text{by Dirichlet's test}}{\longrightarrow} \text{converges}$$

b) $\displaystyle\sum_{n=3}^{\infty} \frac{\sin(8n-2)}{\sqrt{\ln n}} \arctan 2^n$.

$$\sum_{n=3}^{\infty} \frac{\sin(8n-2)}{\sqrt{\ln n}} \longrightarrow \text{converges}$$

$$2^n \uparrow \implies \arctan 2^n \uparrow \text{ and } \arctan 2^n \leq \frac{\pi}{2}$$

$$\therefore \sum_{n=3}^{\infty} \frac{\sin(8n-2)}{\sqrt{\ln n}} \arctan 2^n \overset{\text{by Abel's test}}{\longrightarrow} \text{converges}$$

Exercise 2 (60 points).

Find the radius of convergence and the region of convergence (i.e., additionally study the convergence at the endpoints) of the following power series:

a) $\sum_{n=0}^{\infty}(-1)^n(3n+1)x^n$;

$$R = \lim_{n\to\infty}\left|\frac{(-1)^n(3n+1)}{(-1)^{n+1}(3n+4)}\right| = \lim_{n\to\infty}\left|\frac{(3n+1)}{(3n+4)}\right| = 1$$

$$\sum_{n=0}^{\infty}(-1)^n(3n+1) \longrightarrow \text{diverges}, \sum_{n=0}^{\infty}(-1)^n(3n+1)(-1)^n \longrightarrow \text{diverges} \implies x\in(-1,1)$$

b) $\sum_{n=1}^{\infty}\frac{3^n\cdot x^n}{n+3\cdot 6^n}$;

$$R = \lim_{n\to\infty}\frac{1}{\sqrt[n]{\left|\frac{3^n}{n+3\cdot 6^n}\right|}} = \lim_{n\to\infty}\left|\frac{6\cdot(\frac{n}{6^n}+3)^{\frac{1}{n}}}{3}\right| = 2$$

$$\sum_{n=1}^{\infty}\frac{3^n\cdot 2^n}{n+3\cdot 6^n} \overset{\text{term test}}{\longrightarrow} \text{diverges}, \sum_{n=1}^{\infty}\frac{3^n\cdot(-2)^n}{n+3\cdot 6^n} \overset{\text{term test}}{\longrightarrow} \text{diverges} \implies x\in(-2,2)$$

c) $\sum_{n=1}^{\infty}(-1)^n\frac{8^n x^{3n}}{2n^3+n}$;

$$R_{x^3} = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{8^n}{2n^3+n}}{(-1)^n\frac{8^(n+1)}{2(n+1)^3+(n+1)}}\right| = \frac{1}{8} \implies R = \frac{1}{2}$$

$$\sum_{n=1}^{\infty}(-1)^n\frac{8^n\left(\frac{1}{2}\right)^{3n}}{2n^3+n} \overset{\text{AST}}{\longrightarrow} \text{converges}, \sum_{n=1}^{\infty}(-1)^n\frac{8^n\left(-\frac{1}{2}\right)^{3n}}{2n^3+n} \overset{3>1}{\longrightarrow} \text{converges} \implies x\in\left[-\frac{1}{2},\frac{1}{2}\right]$$

d) $\sum_{n=1}^{\infty}(-1)^n\frac{(x-1)^n}{(3\sqrt{n}+1)\cdot 3^n}$;

$$R = \lim_{n\to\infty}\left|\frac{1}{\sqrt[n]{\frac{1}{(3\sqrt{n}+1)\cdot 3^n}}}\right| = \lim_{n\to\infty}\left|3\sqrt[n]{(3\sqrt{n}+1)}\right| = 3$$

$$\sum_{n=1}^{\infty}(-1)^n\frac{(-3)^n}{(3\sqrt{n}+1)\cdot 3^n} \overset{\text{term test}}{\longrightarrow} \text{diverges}, \sum_{n=1}^{\infty}(-1)^n\frac{3^n}{(3\sqrt{n}+1)\cdot 3^n} \overset{\text{term test}}{\longrightarrow} \text{diverges} \implies x\in(-2,4)$$

e) $\sum_{n=1}^{\infty}\frac{(x-3)^{5n}}{1\cdot 6\cdot \ldots\cdot(5n-4)}$;

$$R_{(x-3)^5} = \lim_{n\to\infty}\left|\frac{\frac{1}{1\cdot 6\cdot\ldots\cdot(5n-4)}}{\frac{1}{1\cdot 6\cdot\ldots\cdot(5n-4))\cdot(5n+1)}}\right| = \lim_{n\to\infty}|5n+1| = +\infty \implies \text{converges } \forall x\in\mathbb{R}$$

f) $\displaystyle\sum_{n=2}^{\infty} \frac{(3\ln n - 1)^n \cdot x^n}{7^n}$.

$$R = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\left|\frac{(3\ln n-1)^n}{7^n}\right|}} = \lim_{n\to\infty} \left|\frac{7}{3\ln n - 1}\right| = 0$$

Exercise 3 (20 points).

a) How many terms of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n \cdot n!}$ is enough to add in order to find the sum correct to within an error of 10^{-5}?

$$|S - S_n| \leq a_{n+1} \leq 10^{-5} \implies \frac{1}{3^n \cdot n!} \leq 10^{-5} \implies 3^n \cdot n! \geq 10^5 \implies n = 4$$

b) How many terms of the series $\displaystyle\sum_{n=1}^{\infty} \frac{6n^5}{(n^6+1)^{11}}$ is enough to add in order to find the sum correct to within an error of 10^{-31}?

$$|S - S_n| \leq \int_1^{\infty} \frac{6x^5}{(x^6+1)^{11}} \,\mathrm{d}x \leq 10^{-31}$$

$$\int_n^{\infty} \frac{6x^5}{(x^6+1)^{11}} \overset{u=x^6+1}{=} \int_{n^6+1}^{\infty} \frac{1}{u^{11}} \,\mathrm{d}u = \left[-\frac{1}{10u^{10}}\right]_{n^6+1}^{\infty} = \frac{1}{10(n^6+1)^{10}} \leq 10^{-31}$$

$$(n^6+1)^{10} \geq 10^{30} \implies n^6 + 1 \geq 1000 \implies n = 4$$