



Digital version

CS101 Calculus 2 Section C - Homework 12

Mher Saribekyan A09210183

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Exercise 1 (30 points).

- a) Use term by term differentiation theorem to find a power series representation for $f(x) = \frac{1}{(3-2x)^2}$, centered at 0. What is the radius of convergence?

$$f_1(x) = \frac{1}{3-2x} \implies f_1'(x) = \frac{2}{(3-2x)^2} \text{ and } f_1(x) = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}x\right)^n = \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} x^n$$

$$\frac{1}{2}f_1'(x) = f(x) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{2^n}{3^{n+1}} nx^{n-1} = \sum_{n=0}^{\infty} \frac{2^n}{3^{n+2}} (n+1)x^n$$

$$R = \lim_{n \rightarrow \infty} \left| \frac{\frac{2^n}{3^{n+2}}(n+1)}{\frac{2^{n+1}}{3^{n+3}}(n+2)} \right| = \lim_{n \rightarrow \infty} \left| \frac{3n+3}{2n+4} \right| = \frac{3}{2}$$

- b) Use part (a) to find a power series for $g(x) = \frac{1}{(3-2x)^3}$.

$$f(x) = \sum_{n=0}^{\infty} \frac{2^n}{3^{n+2}} (n+1)x^n, f'(x) = \frac{4}{(3-2x)^3} = \sum_{n=1}^{\infty} \frac{2^n}{3^{n+2}} (n+1)nx^{n-1} = \sum_{n=0}^{\infty} \frac{2^{n+1}}{3^{n+3}} (n+2)(n+1)x^n$$

$$\therefore g(x) = \sum_{n=0}^{\infty} \frac{2^{n-1}}{3^{n+3}} (n+2)(n+1)x^n$$

- c) Use part (b) to find a power series for $h(x) = \left(\frac{x^2}{3-2x^6}\right)^3$.

$$h(x) = \frac{x^6}{(3-2x^6)^3} = x^6 \cdot \frac{1}{(3-2(x^6))^3} = x^6 \sum_{n=0}^{\infty} \frac{2^{n-1}}{3^{n+3}} (n+2)(n+1)x^{6n}$$

$$\therefore h(x) = \sum_{n=0}^{\infty} \frac{2^{n-1}}{3^{n+3}} (n+2)(n+1)x^{6n+6}$$

Exercise 2 (30 points).

- a) Use term by term integration theorem to find a power series representation for $f(x) = \arctan 3x^2$, centered at 0. What is the radius of convergence?

$$f'(x) = \frac{6x}{1+9x^4} = 6x \cdot \frac{1}{1-(-9x^4)} = 6x \sum_{n=0}^{\infty} (-9x^4)^n = \sum_{n=0}^{\infty} (-1)^n 6 \cdot 9^n x^{4n+1}$$

$$\int f'(x) dx = \sum_{n=0}^{\infty} (-1)^n 6 \cdot 9^n \frac{x^{4n+2}}{4n+2} + c, f(0) = 0 \implies c = 0$$

$$\therefore f(x) = \sum_{n=0}^{\infty} (-1)^n 6 \cdot 9^n \frac{x^{4n+2}}{4n+2}$$

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} 6 \cdot 9^{n+1} \frac{x^{4n+6}}{4n+6}}{(-1)^n 6 \cdot 9^n \frac{x^{4n+2}}{4n+2}} \right| = \lim_{n \rightarrow \infty} \left| x^4 \frac{9(4n+2)}{4n+6} \right| = 9x^4 < 1 \implies |x| < \frac{1}{\sqrt{3}} \implies R = \frac{1}{\sqrt{3}}$$

- b) Use part (a) to evaluate $\int \frac{\arctan 3x^2}{x} dx$, as a power series centered at 0.

$$\frac{\arctan 3x^2}{x} = \frac{f(x)}{x} = \sum_{n=0}^{\infty} (-1)^n 6 \cdot 9^n \frac{x^{4n+1}}{4n+2} \implies \int \frac{\arctan 3x^2}{x} dx = \sum_{n=0}^{\infty} (-1)^n 6 \cdot 9^n \frac{x^{4n+2}}{(4n+2)^2} + c$$

- c) Approximate $\int_0^{1/3} \frac{\arctan 3x^2}{x} dx$, correct to within 10^{-6} .

$$\int_0^{1/3} \frac{\arctan 3x^2}{x} dx = \sum_{n=0}^{\infty} (-1)^n 6 \cdot 9^n \frac{\left(\frac{1}{3}\right)^{4n+2}}{(4n+2)^2} = \sum_{n=0}^{\infty} \frac{6(-1)^n}{3^{2n+2}(4n+2)^2}$$

$$\frac{6}{3^{2n+2}(4n+2)^2} \downarrow \text{ and as } n \rightarrow \infty, \frac{6}{3^{2n+2}(4n+2)^2} \rightarrow 0$$

$$R = |S - S_n| \leq a_{n+1} = \frac{6}{3^{2n+4}(4n+6)^2} \leq 10^{-6} \implies n \geq 4$$

$$\therefore \int_0^{1/3} \frac{\arctan 3x^2}{x} dx \approx \sum_{n=0}^4 \frac{6(-1)^n}{3^{2n+2}(4n+2)^2} = \frac{6}{3^2(2)^2} - \frac{6}{3^4(6)^2} + \frac{6}{3^6(10)^2} - \frac{6}{3^8(14)^2} + \frac{6}{3^{10}(18)^2}$$

Exercise 3 (20 points).

Find the Taylor series at 0 of the following functions:

- a) $f(x) = \sin 2x^4$,

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\sin 2x^4 = \sum_{n=0}^{\infty} (-1)^n \frac{(2x^4)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n 2^{2n+1} \frac{x^{8n+4}}{(2n+1)!}$$

b) $f(x) = \ln(5 + 3x^3)$,

$$\ln(1+x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$\ln(5 + 3x^3) = \ln(5) + \ln\left(1 + \frac{3}{5}x^3\right) = \ln(5) + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{\left(\frac{3}{5}x^3\right)^n}{n}$$

c) $\cos(2x) \cdot \cos(6x)$.

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\begin{aligned} \cos(2x) \cdot \cos(6x) &= \frac{\cos(4x) + \cos(8x)}{2} = \frac{1}{2} \left(\sum_{n=0}^{\infty} (-1)^n \frac{(4x)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} (-1)^n \frac{(8x)^{2n}}{(2n)!} \right) \\ &= \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \frac{(4^{2n} + 8^{2n})x^{2n}}{(2n)!} \end{aligned}$$

Exercise 4 (20 points).

Approximate $\int_{-1}^0 \frac{e^x - 1}{x} dx$, correct to within 10^{-2} .

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1 \implies \text{the integral is not improper and converges}$$

$$\frac{e^x - 1}{x} = x^{-1} \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} - 1 \right) = \frac{1}{x} \sum_{n=1}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!}$$

$$\int_{-1}^0 \frac{e^x - 1}{x} dx = \int_{-1}^0 \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} dx = \sum_{n=0}^{\infty} \int_{-1}^0 \frac{x^n}{(n+1)!} dx = - \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)!(n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!(n+1)}$$

$$\frac{1}{(n+1)!(n+1)} \downarrow \text{ and as } n \rightarrow \infty, \frac{1}{(n+1)!(n+1)} \rightarrow 0$$

$$R = |S - S_n| \leq a_{n+1} = \frac{1}{(n+2)!(n+2)} \leq 10^{-2} \implies n \geq 3$$

$$\therefore \int_{-1}^0 \frac{e^x - 1}{x} dx \approx \sum_{n=0}^3 \frac{(-1)^n}{(n+1)!(n+1)} = \frac{1}{(1)!(1)} - \frac{1}{(2)!(2)} + \frac{1}{(3)!(3)} - \frac{1}{4!(4)} = \frac{229}{288}$$