Digital version

CS101 Caluclus 2 Section C - Homework 1

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Exercise 1 (30 points).

(a) (20 points) Let $f(x) = 8x^3 - 4x, x \in [0, 3]$. Calculate the Riemann sums R_n for f on $[0, 3]$ by dividing the interval into n equal length subintervals and taking right endpoints as sample points

$$\Delta x_i = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

Since we are taking right endpoints as sample points

$$c_i = x_i = a + \frac{b-a}{n} i = 0 + \frac{3}{n} i = \frac{3i}{n}$$

$$\begin{aligned}
R_n &= \sum_{i=1}^{n} f(c_i) \Delta x_i = \sum_{i=1}^{n} f\left(\frac{3i}{n}\right) \frac{3}{n} \\
&= \sum_{i=1}^{n} \left(8\left(\frac{3i}{n}\right)^3 - 4\left(\frac{3i}{n}\right)\right) \frac{3}{n} \\
&= \frac{648}{n^4} \sum_{i=1}^{n} i^3 - \frac{36}{n^2} \sum_{i=1}^{n} i \\
&= \frac{648}{n^4} \left(\frac{n(n+1)}{2}\right)^2 - \frac{36}{n^2} \frac{n(n+1)}{2} \\
&= 162\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) - 18\left(1 + \frac{1}{n}\right)
\end{aligned}$$

$$R_n = 144 + \frac{306}{n} + \frac{162}{n^2}$$

(b) (10 points) Find the limit of R_n.

$$\lim_{n \to \infty} R_n = \lim_{n \to \infty} \left(144 + \frac{306}{n} + \frac{162}{n^2}\right)$$

$$\because \lim_{n \to \infty} \frac{1}{n} = 0 \text{ and } \lim_{n \to \infty} \frac{1}{n^2} = 0$$

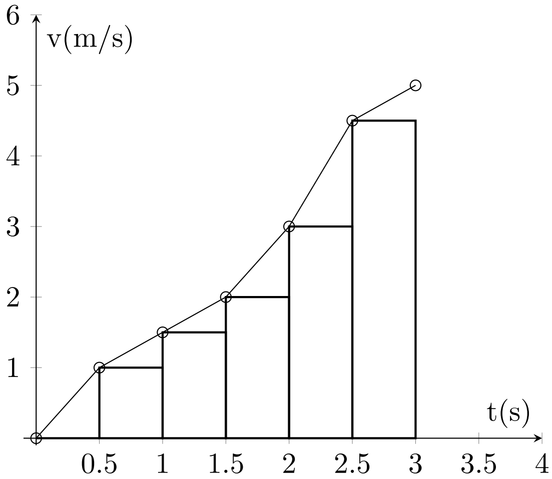
$$\lim_{n \to \infty} R_n = 144$$

Exercise 2 (15 points).

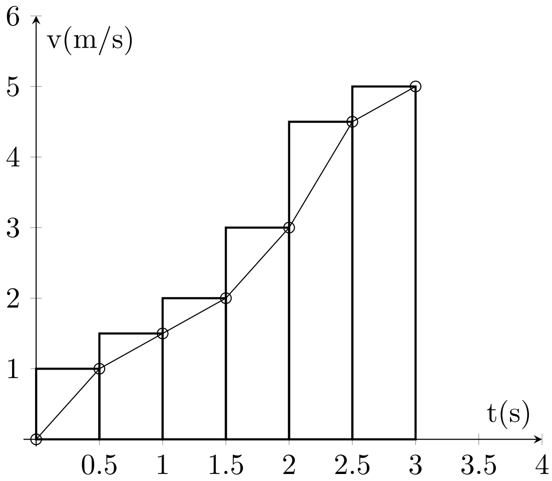
The speed of a runner increased steadily during the first three seconds of a race. Her speed at half-second intervals is given in the table. Find lower and upper estimates for the distance that she travelled during these three seconds.

t(s)	0	0.5	1.0	1.5	2.0	2.5	3.0
v(m/s)	0	1	1.5	2	3	4.5	5

We can get an idea of the distance the runner traveled by graphing time an velocity on x an y axes. The distance the runner travelled equals to velocity multiplied by the time interval. To estimate the distance, we can find the area under the curve. Now we can choose right and left endpoints to calculate the lower and upper estimates of the distance travelled by the runner.



Lower estimates



Upper estimates

$$L = 0 \cdot 0.5 + 1 \cdot 0.5 + 1.5 \cdot 0.5 + 2 \cdot 0.5 + 3 \cdot 0.5 + 4.5 \cdot 0.5 = 6$$

$$U = 1 \cdot 0.5 + 1.5 \cdot 0.5 + 2 \cdot 0.5 + 3 \cdot 0.5 + 4.5 \cdot 0.5 + 5 \cdot 0.5 = 8.5$$

Exercise 3 (25 points).

Let f be an increasing function on [a, b], and let L_n and U_n be the lower and upper sums for n equal length subintervals. Show that

$$U_n - L_n = \frac{b-a}{n}\left(f(b) - f(a)\right)$$

and that f is integrable on $[a,b]$.

First we find $\Delta x_i = \frac{b-a}{n}$.
By the definition of lower sums:

$$L_n = \sum_{i=1}^{n} m_i \Delta x_i, \text{ where } m_i = inf\left(f(x_i), f(x_{i-1})\right)$$

$$\because f \uparrow \implies f(x_i) > f(x_{i-1}) \implies m_i = f(x_{i-1})$$

$$\therefore L_n = \sum_{i=1}^{n} f(x_{i-1}) \frac{b-a}{n}$$

By the definition of upper sums:

$$U_n = \sum_{i=1}^{n} M_i \Delta x_i, \text{ where } M_i = sup\left(f(x_i), f(x_{i-1})\right)$$

$$\because f \uparrow \implies f(x_i) > f(x_{i-1}) \implies M_i = f(x_i)$$

$$\therefore U_n = \sum_{i=1}^{n} f(x_i) \frac{b-a}{n}$$

Now we can calculate the difference of U_n and L_n:

$$U_n - L_n = \frac{b-a}{n}\left(\sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n} f(x_{i-1})\right)$$

$$U_n - L_n = \frac{b-a}{n}\left(f(x_n) + \sum_{i=1}^{n-1} f(x_i) - \sum_{i=2}^{n} f(x_{i-1}) - f(x_0)\right), \text{ let } j = i-1$$

$$= \frac{b-a}{n}\left(f(x_n) - f(x_0) + \sum_{i=1}^{n-1} f(x_i) - \sum_{j=1}^{n-1} f(x_j)\right)$$

$$U_n - L_n = \frac{b-a}{n}\left(f(b) - f(a)\right)$$

$$\because \lim_{n\to\infty} \frac{1}{n} = 0 \text{ and a, b, f(a), f(b) are constants and } b-a \neq 0, f \uparrow \implies f(b) - f(a) \neq 0$$

$$\lim_{n\to\infty} U_n - L_n = 0 \implies \lim_{n\to\infty} U_n = \lim_{n\to\infty} L_n$$

$$\therefore \text{f is integrable on } [a,b]$$

Exercise 4 (30 points).

Express the limits as a definite integral over the given intervals:

(a) (15 points) $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}cos\left(3+\frac{2i}{n}\right)$, [1,3].

$$\Delta x_i = \frac{3-1}{n} = \frac{2}{n}$$

$$\text{Take } c_i = x_i = 1 + \frac{2i}{n}$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}cos\left(3+\frac{2i}{n}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{2}{n}cos\left(2+\left(1+\frac{2i}{n}\right)\right)$$

$$\text{Take } f(x) = cos(2+x)$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\Delta x_i f(c_i)$$

$$= \int_1^3 cos(2+x)dx$$

(b) (15 points) $\lim_{n\to\infty}\left(\frac{n^3}{3n^4+1^4}+\frac{n^3}{3n^4+2^4}+...+\frac{n^3}{3n^4+n^4}\right)$, [0,1].

$$\Delta x_i = \frac{1-0}{n} = \frac{1}{n}$$

$$\text{Take } c_i = x_i = \frac{i}{n}$$

$$\lim_{n\to\infty}\left(\frac{n^3}{3n^4+1^4}+\frac{n^3}{3n^4+2^4}+...+\frac{n^3}{3n^4+n^4}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{n^3}{3n^4+i^4} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\frac{n^4}{3n^4+i^4}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\frac{1}{3+\frac{i}{n}^4}$$

$$\text{Take } f(x) = \frac{1}{3+x^4}$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\Delta x_i f(c_i)$$

$$= \int_0^1 \frac{1}{3+x^4}dx$$