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CS101 Calculus 2 Section C - Homework 2

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February 1, 2024

Exercise 1 (20 points).

Prove the following inequalities:

a) $e^x \geq x + 1,$ for $x \in \mathbb{R}$

$$\text{Take the function } f(x) = e^x - x - 1, x \in \mathbb{R}, f(0) = e^0 - x - 1 = 0$$

$$f'(x) = e^x - 1, \text{ take } e^x - 1 = 0 \implies x = 0, f''(x) = e^x, f''(0) > 0$$

$$x = 0 \text{ is the minimum point of f(x) where } f(x) = 0 \implies e^x - x - 1 \geq 0$$

$$\therefore e^x \geq x + 1, x \in \mathbb{R}$$

b) $\int_0^1 (e^{\sqrt{x}} - 1)^2 dx \geq \frac{1}{2}$

$$\text{From point a) we have } e^u \geq u + 1, u \in \mathbb{R}$$

$$\text{Since } x \geq 0, \text{ we can substitute } u = \sqrt{x} \implies e^{\sqrt{x}} \geq \sqrt{x} + 1 \implies e^{\sqrt{x}} - 1 \geq \sqrt{x}$$

$$\text{Square both sides:} (e^{\sqrt{x}} - 1)^2 \geq x$$

$$\text{Using the property of definite integrals: } \int_0^1 (e^{\sqrt{x}} - 1)^2 dx \geq \int_0^1 x dx = \frac{1}{2}$$

$$\therefore \int_0^1 (e^{\sqrt{x}} - 1)^2 dx \geq \frac{1}{2}$$

c) $0 \leq \int_0^{\frac{\pi}{4}} \frac{\sin^2 x}{\sqrt{4+\cos^5 2x}} dx \leq \frac{\pi}{16}$

$$0 \leq x \leq \frac{\pi}{4} \implies 0 \leq \sin x \leq \frac{1}{\sqrt{2}} \implies 0 \leq \sin^2 x \leq \frac{1}{2}$$

$$0 \leq x \leq \frac{\pi}{4} \implies \frac{1}{\sqrt{2}} \leq \cos x \leq 1 \implies \cos x \geq 0 \implies \cos^5(2x) \geq 0$$

$$\implies 4 + \cos^5(2x) \geq 4 \implies \sqrt{4 + \cos^5(2x)} \geq \sqrt{4} = 2$$

$$0 \leq \frac{\sin^2 x}{\sqrt{4 + \cos^5 2x}} \leq \frac{\frac{1}{2}}{2} \implies 0 \leq \frac{\sin^2 x}{\sqrt{4 + \cos^5 2x}} \leq \frac{1}{4}$$

$$\text{Using the property of definite integrals: } \int_0^{\frac{\pi}{4}} 0 dx \leq \int_0^{\frac{\pi}{4}} \frac{\sin^2 x}{\sqrt{4 + \cos^5 2x}} dx \leq \int_0^{\frac{\pi}{4}} \frac{1}{4} dx = \frac{\pi}{4} \cdot \frac{1}{4} = \frac{\pi}{16}$$

$$\therefore 0 \leq \int_0^{\frac{\pi}{4}} \frac{\sin^2 x}{\sqrt{4 + \cos^5 2x}} dx \leq \frac{\pi}{16}$$

d) $\int_1^5 \frac{2\arctan x}{1+\ln^2 x}dx \le 4\pi$

$$\arctan x \le \frac{\pi}{2} \implies 2\arctan x \le \pi$$

$$\ln^2 x \ge 0 \implies 1+\ln^2 x \ge 1 \implies \frac{2\arctan x}{1+\ln^2 x} \le \pi$$

$$\text{Using the property of definite integrals: } \int_1^5 \frac{2\arctan x}{1+\ln^2 x}dx \le \int_1^5 \pi dx = (5-1)\cdot\pi = 4\pi$$

$$\therefore \int_1^5 \frac{2\arctan x}{1+\ln^2 x}dx \le 4\pi$$

Exercise 2 (30 points).

Expressing integrals in terms of areas of some regions find the average value f_{avg} of the function $f(x)$ on $[a,b]$ and all points $c \in [a,b]$ such that $f(c) = f_{avg}$, where

a) $f(x) = 2|x-1|$ and $[a,b] = [-1,3]$

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{3-(-1)}\int_{-1}^3 2|x-1|dx$$

$$= \frac{1}{4}\left(\int_{-1}^1 2(1-x)dx + \int_1^3 2(x-1)dx\right) = \frac{1}{2}\left(\int_{-1}^1 (1-x)dx + \int_1^3 (x-1)dx\right)$$

$$\text{Evaluate areas of two rectangles: } f_{avg} = \frac{1}{2}\left(\frac{2\cdot 2}{2} + \frac{2\cdot 2}{2}\right) = 2$$

$$\text{Take } c \in [a,b] \text{ such that } f(c) = f_{avg} = 2 \implies 2|c-1| = 2 \implies |c-1| = 1$$

$$\implies \begin{cases} c-1 = 1 \\ c-1 = -1 \end{cases} \implies \begin{cases} c = 2 \\ c = 0 \end{cases}$$

b) $f(x) = x^2\sin x$ and $[a,b] = [-1,1]$

$$x^2 = (-x)^2 \text{ and } \sin(x) = -\sin(-x) \implies f(-x) = (-x)^2\sin(-2) = -x^2\sin(x) = -f(x)$$

$$\implies f(-x) = -f(x) \implies f(x) = -f(-x)$$

$$\int_{-a}^0 f(x)dx = \int_{-a}^0 -f(-x)dx = \int_{-a}^0 f(-x)d(-x) \overset{u=-x}{=} \int_a^0 f(u)du$$

$$= -\int_0^a f(u)du = -\int_0^a f(x)dx$$

$$\int_a^b f(x)dx = \int_{-1}^0 f(x)dx + \int_0^1 f(x)dx = -\int_0^1 f(x)dx + \int_0^1 f(x)dx = 0$$

$$\therefore f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{2}\cdot 0 = 0$$

$$\text{Take } c \in [a,b] \text{ such that } f(c) = f_{avg} = 0 \implies c^2\sin c = 0 \implies c = 0$$

c) $f(x) = \sqrt{9 - x^2}$ and $[a, b] = [-3, 0]$

$$y := \sqrt{9 - x^2} \implies x^2 + y^2 = 3^2, y \geq 0, x \in [-3, 0]$$

$$\text{We have a quarter circle with radius 3 centered at the point (0,0)} \implies \text{Area} = \frac{\pi r^2}{4} = \frac{9\pi}{4}$$

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x)dx = \frac{1}{b-a} \cdot \text{Area} = \frac{1}{3} \cdot \frac{9\pi}{4} = \frac{3\pi}{4}$$

$$\text{Take } c \in [a, b] \text{ such that } f(c) = f_{avg} = \frac{3\pi}{4} \implies \sqrt{9 - c^2} = \frac{3\pi}{4} \implies c^2 = 9 - \frac{9\pi}{16}$$

$$\implies c^2 = \frac{9}{16}(16 - \pi) \implies c = -\frac{3}{4}\sqrt{16 - \pi}, \text{ since } c \in [-3, 0]$$

Exercise 3 (50 points).

Calculate.

a) $f(1), f'(x), f'(4), f'(3x), (f(3x))'$, if $f(x) = \int_1^x \sqrt[3]{t(3t+4)}dt, x \geq 1$

$$f(1) = \int_1^1 \sqrt[3]{t(3t+4)}dt = 0$$

$$f'(x) = \frac{d}{dx} \int_1^x \sqrt[3]{t(3t+4)}dt \overset{\text{FTC}}{=} \sqrt[3]{x(3x+4)}$$

$$f'(4) = \sqrt[3]{4(3 \cdot 4 + 4)} = \sqrt[3]{64} = 4$$

$$f'(3x) = \sqrt[3]{3x(3 \cdot 3x + 4)} = \sqrt[3]{3x(9x+4)}$$

$$(f(3x))' = \frac{d}{dx} \int_1^{3x} \sqrt[3]{t(3t+4)}dt \overset{\text{FTC}}{=} (3x)' \cdot \sqrt[3]{3x(3 \cdot 3x + 4)} = 3\sqrt[3]{3x(9x+4)}$$

b) $\frac{d}{dx} \int_{3x+2}^{4x^2} x^3 \cdot (t - \ln t)dt$

$$\frac{d}{dx} \int_{3x+2}^{4x^2} x^3 \cdot (t - \ln t)dt = \frac{d}{dx} \left(x^3 \cdot \int_{3x+2}^{4x^2} (t - \ln t)dt \right), \text{ use product rule}$$

$$= \frac{d}{dx}(x^3) \cdot \int_{3x+2}^{4x^2} (t - \ln t)dt + x^3 \cdot \frac{d}{dx} \int_{3x+2}^{4x^2} (t - \ln t)dt$$

$$\overset{\text{FTC}}{=} 3x^2 \cdot \int_{3x+2}^{4x^2} (t - \ln t)dt + x^3 \cdot \left((4x^2 - \ln(4x^2)) \cdot (4x^2)' - (3x + 2 - \ln(3x+2) \cdot (3x+2)' \right)$$

$$= 3x^2 \cdot \int_{3x+2}^{4x^2} (t - \ln t)dt + x^3 \left(8x(4x^2 - 2\ln(2x)) - 3(3x + 2 - \ln(3x+2)) \right)$$

$$= 3x^2 \cdot \int_{3x+2}^{4x^2} (t - \ln t)dt + x^3 \left(32x^3 - 16\ln(2x) - 9x - 6 + 3\ln(3x+2) \right)$$

$$= 3x^2 \cdot \int_{3x+2}^{4x^2} (t - \ln t)dt + 32x^6 - 9x^4 - 6x^3 - 16x^3\ln(2x) + 3x^3\ln(3x+2)$$

c) $\frac{d}{dx}\ln\Big(\int_{\sin x}^{1+x^2}\sqrt[3]{t+e^t}dt\Big)$

$$\frac{d}{dx}\ln\left(\int_{\sin x}^{1+x^2}\sqrt[3]{t+e^t}dt\right) = \ln'\left(\int_{\sin x}^{1+x^2}\sqrt[3]{t+e^t}dt\right)\cdot\frac{d}{dx}\int_{\sin x}^{1+x^2}\sqrt[3]{t+e^t}dt$$

$$\stackrel{\text{FTC}}{=} \frac{\sqrt[3]{1+x^2+e^{1+x^2}}\cdot(1+x^2)' - \sqrt[3]{\sin x+e^{\sin x}}\cdot(\sin x)'}{\int_{\sin x}^{1+x^2}\sqrt[3]{t+e^t}dt}$$

$$= \frac{2x\sqrt[3]{1+x^2+e^{1+x^2}} - \cos x\sqrt[3]{\sin x+e^{\sin x}}}{\int_{\sin x}^{1+x^2}\sqrt[3]{t+e^t}dt}$$

d) $\frac{d}{dx}\left(\int_{3x}^{\ln x}\tan\big(1+5t^3\big)dt\right)^3$

$$\frac{d}{dx}\left(\int_{3x}^{\ln x}\tan\big(1+5t^3\big)dt\right)^3 = 3\left(\int_{3x}^{\ln x}\tan\big(1+5t^3\big)dt\right)^2\cdot\frac{d}{dx}\int_{3x}^{\ln x}\tan\big(1+5t^3\big)dt$$

$$\stackrel{\text{FTC}}{=} 3\left(\int_{3x}^{\ln x}\tan\big(1+5t^3\big)dt\right)^2\cdot\Big(\tan\big(1+5(\ln x)^3\big)\cdot(\ln x)' - \tan\big(1+5(3x)^3\big)\cdot(3x)'\Big)$$

$$= 3\left(\int_{3x}^{\ln x}\tan\big(1+5t^3\big)dt\right)^2\cdot\left(\frac{\tan\big(1+5\ln^3 x\big)}{x} - 3\tan\big(1+135x^3\big)\right)$$

e) Let $f(x)=\int_0^x\sqrt[3]{\cos^3(t^2)-4}dt$, for $x\in\mathbb{R}$. Show that f decreases on $\mathbb{R}$.

$$f'(x) = \frac{d}{dx}\int_0^x\sqrt[3]{\cos^3(t^2)-4}dt \stackrel{\text{FTC}}{=} \sqrt[3]{\cos^3(x^2)-4}$$

$$-1\le\cos x\le 1 \implies -1\le\cos^3(x^2)\le 1 \implies -5\le\cos^3(x^2)-4\le -3$$
$$\implies \sqrt[3]{-5}\le\sqrt[3]{\cos^3(x^2)-4}\le\sqrt[3]{-3} \implies \sqrt[3]{\cos^3(x^2)-4}\le 0\, x\in\mathbb{R}$$
$$f'(x)\le 0, x\in\mathbb{R} \implies f \text{ decreases on } \mathbb{R}$$