Digital version

CS101 Calculus 2 Section C - Homework 3

Mher Saribekyan A09210183

February 8, 2024

Exercise 1 (50 points).

Evaluate

a) $\int_{-4}^{4}(x^3 + 3\sin 4x)\cos(x^3)dx$,

$$\text{Let } f(x) = (x^3 + 3\sin 4x)\cos(x^3)$$
$$f(-x) = ((-x)^3 + 3\sin(4(-x)))\cos((-x)^3) = (-x^3 + (-3\sin(4x)))\cos(-x^3)$$
$$= -(x^3 + 3\sin(4x)\cos(x^3) = -f(x) \implies \text{The function is odd}$$
$$\therefore \int_{-4}^{4}(x^3 + \sin 4x)\cos(x^3)dx = 0$$

b) $\int_{e^{-2}}^{e^2} \frac{\ln x}{x\sqrt[3]{3+(\ln x)^2}}dx$,

$$\int_{e^{-2}}^{e^2} \frac{\ln x}{x\sqrt[3]{3+(\ln x)^2}}dx = \int_{e^{-2}}^{e^2} \frac{\ln x}{\sqrt[3]{3+(\ln x)^2}}d(\ln x) \overset{u=\ln x}{=} \int_{-2}^{2} \frac{u}{\sqrt[3]{3+u^2}}du$$
$$\text{Let } f(u) = \frac{u}{\sqrt[3]{3+u^2}}$$
$$f(-u) = \frac{-u}{\sqrt[3]{3+(-u)^2}} = -\frac{u}{\sqrt[3]{3+u^2}} = -f(u) \implies \text{The function is odd}$$
$$\therefore \int_{e^{-2}}^{e^2} \frac{\ln x}{x\sqrt[3]{3+(\ln x)^2}}dx = 0$$

c) $\int_{-\pi}^{\frac{\pi}{2}} x|\sin x|dx$,

$$|\sin x| = \begin{cases} -\sin x, -\pi \leq x \leq 0 \\ \sin x, 0 \leq x \leq \frac{\pi}{2} \end{cases} \implies x|\sin x| = \begin{cases} -x\sin x, -\pi \leq x \leq 0 \\ x\sin x, 0 \leq x \leq \frac{\pi}{2} \end{cases}$$
$$\int_{-\pi}^{\frac{\pi}{2}} x|\sin x|dx = \int_{-\pi}^{0} -x\sin x dx + \int_{0}^{\frac{\pi}{2}} x\sin x dx$$
$$= \int_{0}^{\frac{\pi}{2}} xd(-\cos x) - \int_{-\pi}^{0} xd(-\cos x)$$

$$\overset{\text{IbP}}{=} [-x\cos x]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (x)'(-\cos x)dx - [-x\cos x]_{-\pi}^0 + \int_{-\pi}^0 (x)'(-\cos x)dx$$

$$\overset{\text{FTC}}{=} [-x\cos x + \sin x]_0^{\frac{\pi}{2}} + [x\cos x - \sin x]_{-\pi}^0 = \left(\left(-\frac{\pi}{2}\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - 0\right) + (0 - (-\pi\cos(-\pi) - \sin(-\pi)))$$

$$= 0((0+1)-0) + (0 - (\pi - 0)) = 1 - \pi$$

$$\therefore \int_{-\pi}^{\frac{\pi}{2}} x|\sin x|dx = 1 - \pi$$

d) $\int_{-3}^{11\pi-3} \frac{\cos(4x)\sin^3(2x)}{\sqrt{3-\sin^2 x}}dx$,

$$\text{Let } f(x) = \frac{\cos(4x)\sin^3(2x)}{\sqrt{3-\sin^2 x}}, f(x+\pi) = \frac{\cos(4(x+\pi))\sin^3(2(x+\pi))}{\sqrt{3-\sin^2(x+\pi)}} = \frac{\cos(4x+4\pi)\sin^3(2x+2\pi)}{\sqrt{3-\frac{1-\cos(2x+2\pi)}{2}}}$$

$$= \frac{\cos(4x)\sin^3(2x)}{\sqrt{3-\frac{1-\cos(2x)}{2}}} = \frac{\cos(4x)\sin^3(2x)}{\sqrt{3-\sin^2 x}} = f(x) \implies f \text{ has a period of } \pi$$

$$\int_{-3}^{11\pi-3} \frac{\cos(4x)\sin^3(2x)}{\sqrt{3-\sin^2 x}}dx \overset{T=\pi}{=} 11\int_0^{\pi} \frac{\cos(4x)\sin^3(2x)}{\sqrt{3-\sin^2 x}}dx \overset{T=\pi}{=} 11\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos(4x)\sin^3(2x)}{\sqrt{3-\sin^2 x}}dx$$

$$f(-x) = \frac{\cos(4(-x))\sin^3(2(-x))}{\sqrt{3-\sin^2(-x)}} = \frac{\cos(4x)(-\sin^3(2x))}{\sqrt{3-\sin^2 x}} = -f(x) \implies f \text{ is odd}$$

$$\therefore \int_{-3}^{11\pi-3} \frac{\cos(4x)\sin^3(2x)}{\sqrt{3-\sin^2 x}}dx = 0$$

e) $\int_{-3}^{3} x\sqrt{10-2x}dx$,

$$\int_{-3}^{3} x\sqrt{10-2x}dx = \sqrt{2}\int_{-3}^{3} x(5-x)^{\frac{1}{2}}dx = \sqrt{2}\int_{-3}^{3} xd\left(-\frac{(5-x)^{\frac{3}{2}}}{\frac{3}{2}}\right) = -\frac{2\sqrt{2}}{3}\int_{-3}^{3} xd\left((5-x)^{\frac{3}{2}}\right)$$

$$\overset{\text{IbP}}{=} -\frac{2\sqrt{2}}{3}\left(\left[x(5-x)^{\frac{3}{2}}\right]_{-3}^{3} - \int_{-3}^{3}(x)'(5-x)^{\frac{3}{2}}dx\right) = -\frac{2\sqrt{2}}{3}\left(\left[x(5-x)^{\frac{3}{2}}\right]_{-3}^{3} - \int_{-3}^{3}(5-x)^{\frac{3}{2}}dx\right)$$

$$\overset{\text{FTC}}{=} -\frac{2\sqrt{2}}{3}\left(\left[x(5-x)^{\frac{3}{2}} + \frac{(5-x)^{\frac{5}{2}}}{\frac{5}{2}}\right]_{-3}^{3}\right) = -\frac{2\sqrt{2}}{3}\left(\left(3\cdot 2^{\frac{3}{2}} + \frac{2^{\frac{5}{2}}}{\frac{5}{2}}\right) - \left(-3\cdot 8^{\frac{3}{2}} + \frac{8^{\frac{5}{2}}}{\frac{5}{2}}\right)\right)$$

$$= -\frac{2\sqrt{2}}{3}\left(\left(6\sqrt{2} + \frac{8\sqrt{2}}{5}\right) - \left(-48\sqrt{2} + \frac{256\sqrt{2}}{5}\right)\right) = -\frac{2\sqrt{2}}{3}\cdot\frac{22}{5}\sqrt{2} = -\frac{88}{15}$$

$$\therefore \int_{-3}^{3} x\sqrt{10-2x}dx = -\frac{88}{15}$$

f) $\int_0^6 \left|x^2-5x+6\right|dx$.

$$x^2-5x+6=(x-2)(x-3) \implies \begin{cases} x^2-5x+6 \geq 0, x \leq 2 \text{ or } x \geq 3 \\ x^2-5x+6 < 0, 2 < x < 3 \end{cases}$$

$$\left|x^2-5x+6\right| = \begin{cases} x^2-5x+6, x \in [0,2] \\ -(x^2-5x+6), x \in (2,3) \\ x^2-5x+6, x \in [3,6] \end{cases}$$

$$\int_0^6 \left|x^2-5x+6\right|dx = \int_0^2 x^2-5x+6dx - \int_2^3 x^2-5x+6dx + \int_3^6 x^2-5x+6dx$$

$$\overset{\text{FTC}}{=} \left[\frac{x^3}{3}-\frac{5x^2}{2}+6x\right]_0^2 - \left[\frac{x^3}{3}-\frac{5x^2}{2}+6x\right]_2^3 + \left[\frac{x^3}{3}-\frac{5x^2}{2}+6x\right]_3^6$$

$$= \left(\frac{2^3}{3}-\frac{5\cdot 2^2}{2}+6\cdot 2\right) - \left(\frac{0^3}{3}-\frac{5\cdot 0^2}{2}+6\cdot 0\right) - \left(\frac{3^3}{3}-\frac{5\cdot 3^2}{2}+6\cdot 3\right) + \left(\frac{2^3}{3}-\frac{5\cdot 2^2}{2}+6\cdot 2\right)$$

$$+ \left(\frac{6^3}{3}-\frac{5\cdot 6^2}{2}+6\cdot 6\right) - \left(\frac{3^3}{3}-\frac{5\cdot 3^2}{2}+6\cdot 3\right) = \frac{14}{3} - 0 - \frac{9}{2} + \frac{14}{3} + 18 - \frac{9}{2} = \frac{55}{3}$$

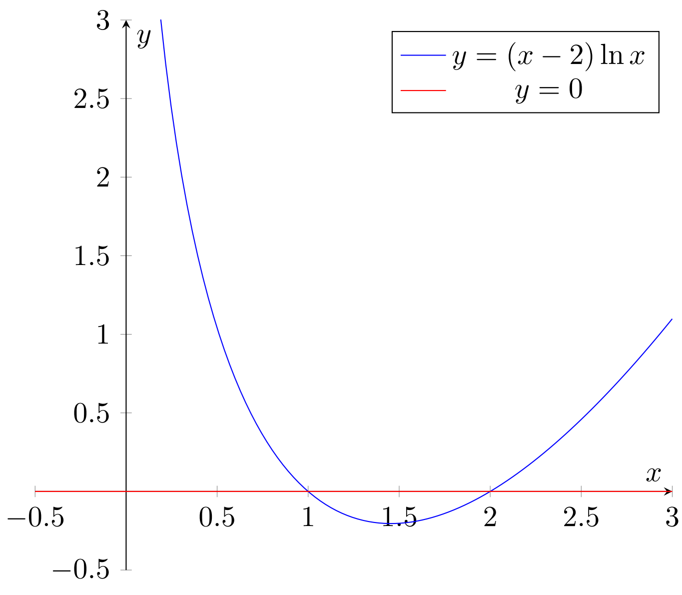
$$\therefore \int_0^6 \left|x^2-5x+6\right|dx = \frac{55}{3}$$

Exercise 2 (50 points).

Sketch the region bounded by given curves and find its area:

a) $y=(x-2)\ln x$ and $y=0$

$$(x-2)\ln x = 0 \implies x=1 \text{ and } x=2 \implies (x-2)\ln x \leq 0, x \in [1,2]$$



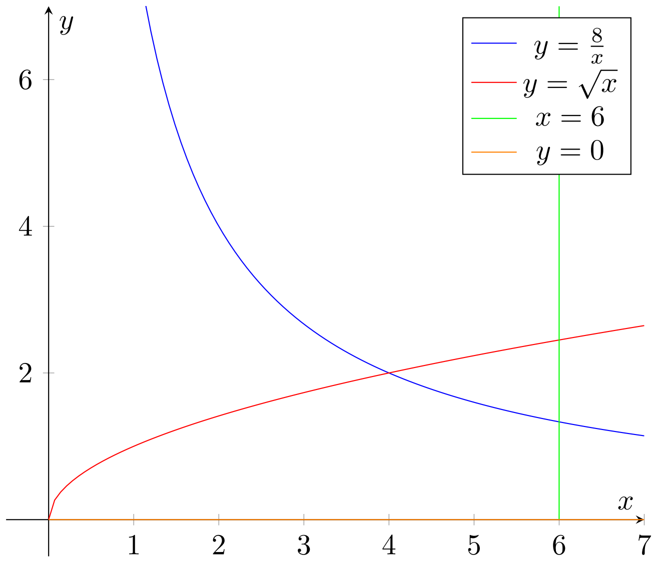
$$Area = \int_1^2 0-(x-2)\ln x dx = \int_1^2 2\ln x dx - \int_1^2 \ln x d\left(\frac{x^2}{2}\right) \overset{\text{IbP}}{=} [2x\ln x - 2x]_1^2 - \left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_1^2$$

$$= ((2\cdot 2\ln 2 - 2\cdot 2) - (2\cdot 1\ln 1 - 2\cdot 1)) - \left(\left(\frac{2^2}{2}\ln 2 - \frac{2^2}{4}\right) - \left(\frac{1^2}{2}\ln 1 - \frac{1^2}{4}\right)\right) = 4\ln 2 - 2 - 2\ln 2 + \frac{3}{4}$$

$$\therefore Area = 2\ln 2 - \frac{5}{4}$$

b) $y = \frac{8}{x}, y = \sqrt{x}, y = 0$ and $x = 6$

$$\text{Take } \frac{8}{x} = \sqrt{x} \implies x^{\frac{3}{2}} = 8 \implies x = 4 \implies \text{they intercept at } (4,2)$$

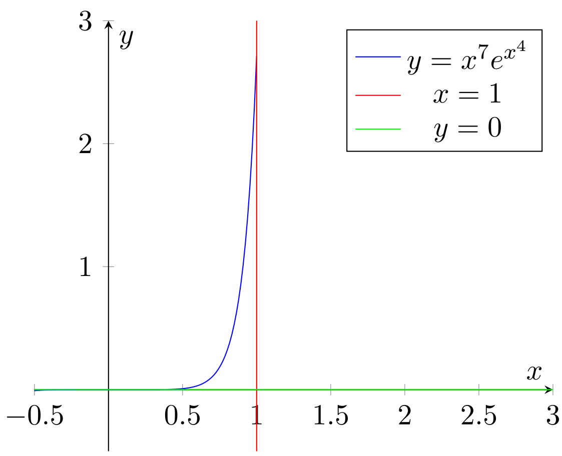


$$Area = \int_0^4 \sqrt{x}dx + \int_4^6 \frac{8}{x}dx \overset{\text{FTC}}{=} \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4 + 8[\ln x]_4^6 = \frac{2}{3}8 + 8\ln 6 - 8\ln 4 = \frac{16}{3} + \ln 3 + \ln 2 - 2\ln 2$$

$$\therefore Area = \frac{16}{3} + \ln 3 - \ln 2$$

c) $y = x^7 e^{x^4}, y = 0, x = 1$

$$0^7 e^{0^4} = 0, 1^7 e^{1^4} = e \text{ and } x \geq 0 \implies x^7 e^{x^4} \geq 0$$



$$Area = \int_0^1 x^7 e^{x^4} dx = \int_0^1 x^3 \cdot x^4 \cdot e^{x^4} dx = \int_0^1 x^4 e^{x^4} d\left(\frac{x^4}{4}\right) = \frac{1}{4}\int_0^1 x^4 e^{x^4} d(x^4)$$

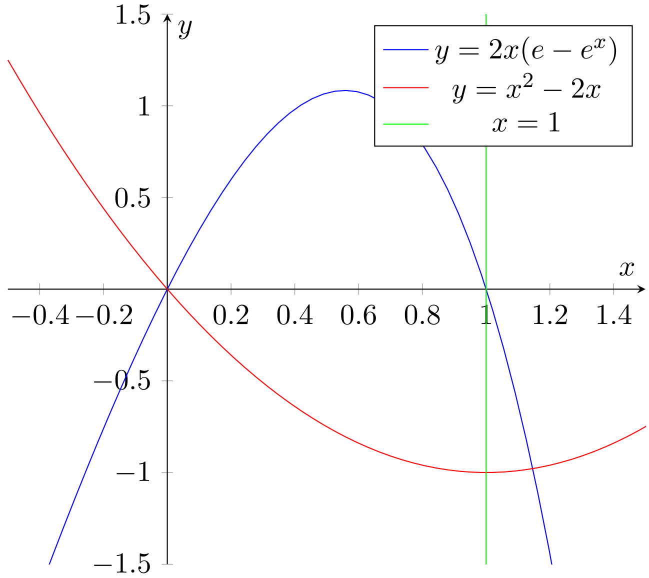
$$\overset{u=x^4}{=} \frac{1}{4}\int_0^1 ue^u du = \frac{1}{4}\int_0^1 ude^u \overset{\text{IbP}}{=} \frac{1}{4}\left([ue^u]_0^1 - \int_0^1 e^u du\right) \overset{\text{FTC}}{=} \frac{1}{4}\left[ue^u - e^u\right]_0^1$$

$$= \frac{1}{4}\left(1e^1 - e^1 - 0e^0 + e^0\right) = \frac{1}{4}$$

$$\therefore Area = \frac{1}{4}$$

d) $y = 2x(e - e^x), y = x^2 - 2x$ and $x \leq 1$

$$2x(e - e^x) = x^2 - 2x \implies x = 0$$



$$Area = \int_0^1 \left(2x(e - e^x) - (x^2 - 2x)\right) dx = \int_0^1 2xedx - \int_0^1 2xe^x dx - \int_0^1 x^2 dx + \int_0^1 2xdx$$

$$\overset{\text{FTC}}{=} \left[\frac{2ex^2}{2} - \frac{x^3}{3} + \frac{2x^2}{2}\right]_0^1 - 2\int_0^1 xde^x \overset{\text{IbP}}{=} \left[ex^2 - \frac{x^3}{3} + x^2 - 2xe^x\right]_0^1 + 2\int_0^1 e^x dx$$

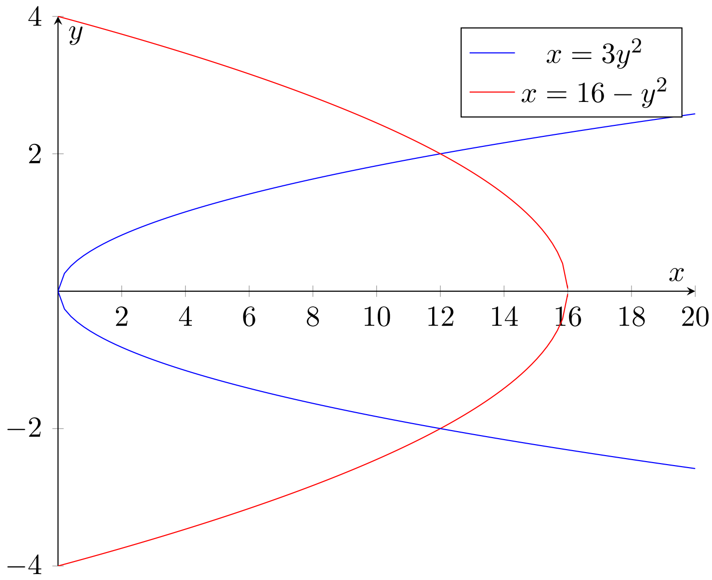
$$\overset{\text{FTC}}{=} \left[ex^2 - \frac{x^3}{3} + x^2 - 2xe^x + 2e^x\right]_0^1 = \left(e1^2 - \frac{1^3}{3} + 1^2 - 2 \cdot 1e^1 + 2e^1\right) - (2e^0) = e - \frac{4}{3}$$

$$\therefore Area = e - \frac{4}{3}$$

e) $x = 3y^2$ and $x = 16 - y^2$

$$3y^2 = 16 - y^2 \implies 4y^2 = 16 \implies y = \pm 2$$

$$x = 3 \cdot 0^2 = 0, x = 16 - 0^2 = 16 \implies 3y^2 \leq 16 - y^2, y \in [-2, 2]$$



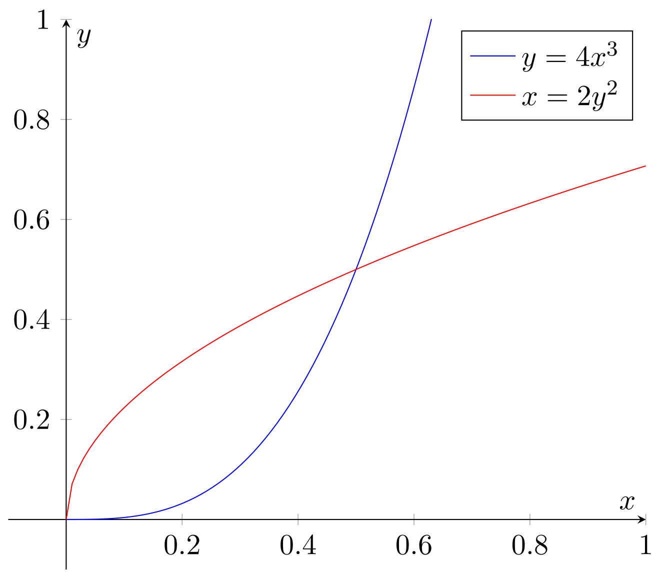
$$Area = \int_{-2}^{2} 16 - y^2 - 3y^2 dy = \int_{-2}^{2} 16 - 4y^2 dy \overset{\text{FTC}}{=} \left[16y - 4\frac{y^3}{3}\right]_{-2}^{2}$$

$$= \left(16 \cdot 2 - 4\frac{2^3}{3}\right) - \left(16 \cdot (-2) - 4\frac{(-2)^3}{3}\right) = \frac{128}{3}$$

$$\therefore Area = \frac{128}{3}$$

f) $y = 4x^3$ and $x = 2y^2$

$$y = 4x^3 \text{ and } x = 2y^2 \implies y = 2^5 y^6 \implies y = 0, y = \frac{1}{2}$$



$$\because x = 2y^2 \text{ and } y \geq 0 \implies y = \sqrt{\frac{x}{2}}$$

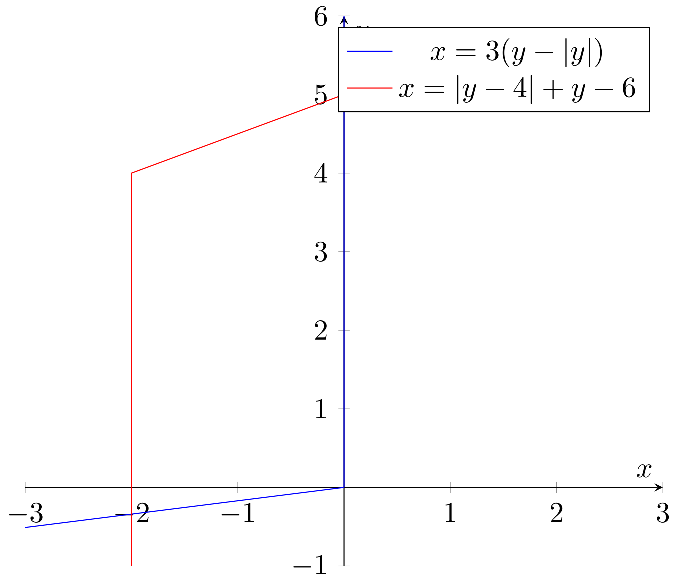
$$Area = \int_0^{\frac{1}{2}} \sqrt{\frac{x}{2}} - 4x^3 dx = \int_0^{\frac{1}{2}} \frac{1}{\sqrt{2}}\sqrt{x} - 4x^3 dx \overset{\text{FTC}}{=} \left[\frac{1}{\sqrt{2}}\frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 4\frac{x^4}{4}\right]_0^{\frac{1}{2}} = \frac{2 \cdot \frac{1}{2}^{\frac{3}{2}}}{3\sqrt{2}} - \left(\frac{1}{2}\right)^4$$

$$\therefore Area = \frac{5}{48}$$

g) $x = 3(y - |y|)$ and $x = |y-4| + y - 6$

$$x = 3(y - |y|) \implies \begin{cases} x = 0, y \geq 0 \\ x = 6y, y \leq 0 \end{cases} \quad \text{and } x = |y-4| + y - 6 \implies \begin{cases} x = 2y - 10, y \geq 4 \\ x = -2, y \leq 4 \end{cases}$$

$$0 = 2y - 10 \implies y = 5 \text{ and } -2 = 6y \implies y = -\frac{1}{3}$$



$$Area = \text{ Areas of rectangle and two triangles } = 2 \cdot 4 + \frac{1 \cdot 2}{2} + + \frac{\frac{1}{3} \cdot 2}{2}$$

$$\therefore Area = \frac{28}{3}$$