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# CS101 Calculus 2 Section C - Homework 4

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## Exercise 1 (40 points).

(a) The base of a solid is the region bounded by the curve $y = x^4 \cos x^3$ for $x \in \left[0, \sqrt[3]{\frac{\pi}{2}}\right]$ and line $y = 0$. Find the volume of that solid given that the cross sections perpendicular to the $x$-axis are triangles with height equal to the base.
Area of a right triangle with equal base and height is $S = \frac{a^2}{2}$, where $a$ is the base of the triangle and it is equal to $f(x)$.

$$\text{Volume} = \int_0^{\sqrt[3]{\frac{\pi}{2}}} \frac{\left(x^4 \cos x^3\right)^2}{2} dx = \frac{1}{2} \int_0^{\sqrt[3]{\frac{\pi}{2}}} x^8 \cos^2 x^3 dx = \frac{1}{2} \int_0^{\sqrt[3]{\frac{\pi}{2}}} x^6 \cos^2 x^3 d\left(\frac{x^3}{3}\right)$$

$$\overset{u=x^3}{=} \frac{1}{6} \int_0^{\frac{\pi}{2}} u^2 \cos^2 u du = \frac{1}{6} \int_0^{\frac{\pi}{2}} \frac{u^2}{2} + \frac{u^2 \cos 2u}{2} u du + \frac{1}{12} \overset{\text{FTC}}{=} \left[\frac{u^3}{36}\right]_0^{\frac{\pi}{2}} + \frac{1}{12} \int_0^{\frac{\pi}{2}} u^2 d\left(\frac{\sin 2u}{2}\right)$$

$$\overset{\text{IbP}}{=} \frac{\pi^3}{288} + \left[\frac{u^2 \sin 2u}{24}\right]_0^{\frac{\pi}{2}} - \frac{1}{12} \int_0^{\frac{\pi}{2}} u \sin 2u du = \frac{\pi^3}{288} - \frac{1}{12} \int_0^{\frac{\pi}{2}} u d\left(\frac{-\cos 2u}{2}\right)$$

$$\overset{\text{IbP}}{=} \frac{\pi^3}{288} + \left[\frac{u \cos 2u}{24}\right]_0^{\frac{\pi}{2}} - \frac{1}{24} \int_0^{\frac{\pi}{2}} \cos 2u du \overset{\text{FTC}}{=} \frac{\pi^3}{288} + \left[\frac{u \cos 2u}{24} - \frac{\sin 2u}{48}\right]_0^{\frac{\pi}{2}} = \frac{\pi^3 - 6\pi}{288}$$

(b) The base of a solid is the region bounded by the curves $y = \sqrt[4]{1 - 9x^2}$, and $y = 0$. Find the volume of that solid given that the cross sections perpendicular to the $x$-axis are rectangles with height twice as big as width.

$$\sqrt[4]{1 - 9x^2} = 0 \implies x \in \left\{-\frac{1}{3}, \frac{1}{3}\right\}$$

Area of a right triangle with height twice as big as width is $S = \frac{2a^2}{2} = a^2$, where $a$ is the base of the triangle and it is equal to $f(x)$.

$$\text{Volume} = \int_{-\frac{1}{3}}^{\frac{1}{3}} \left(\sqrt[4]{1 - 9x^2}\right)^2 dx \overset{x=\frac{1}{3}\sin\varphi}{=} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - 9\left(\frac{1}{3}\sin\varphi\right)^2} d\left(\frac{1}{3}\sin\varphi\right)$$

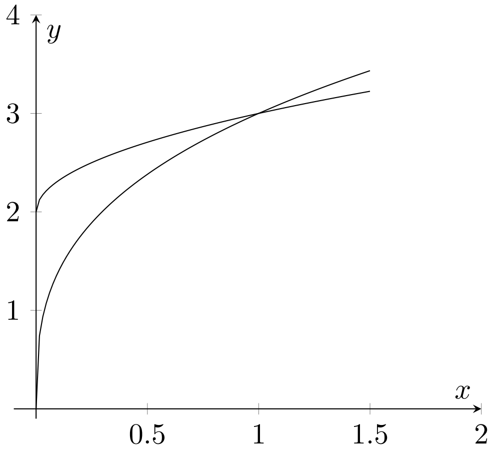
$$= \frac{1}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi \sqrt{1 - \sin^2\varphi} d\varphi = \frac{1}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\varphi d\varphi = \frac{1}{3} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \cos 2\varphi}{2} \varphi d\varphi$$

$$\overset{\text{FTC}}{=} \left[\frac{\varphi}{6} + \frac{\sin 2\varphi}{12}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \left(\frac{\pi}{12} + 0\right) - \left(-\frac{\pi}{12} + 0\right) = \frac{\pi}{6}$$

## Exercise 2 (30 points).

(a) Sketch the region bounded by the curves $y = 3\sqrt[3]{x}$ and $y = \sqrt{x} + 2$, and find the volume of the solid generated by revolving this region about the $x$-axis.

$$3\sqrt[3]{x} = \sqrt{x} + 2 \implies x = 1, y = 3 \text{ and } \sqrt{x} + 2 \geq 3\sqrt[3]{x}, x \in [0,1]$$

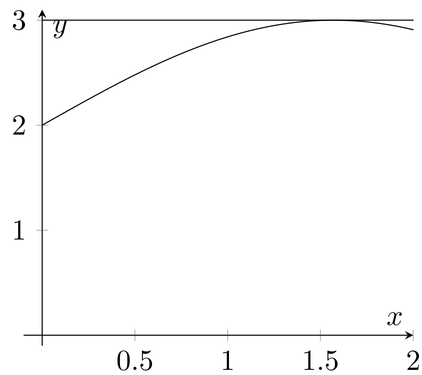



$$\text{Volume} = \pi \int_0^1 \left(\sqrt{x}+2\right)^2 - \left(3\sqrt[3]{x}\right)^2 dx = \pi \int_0^1 x + 4x^{\frac{1}{2}} + 4 - 9x^{\frac{2}{3}} dx$$

$$\overset{\text{FTC}}{=} \left[\frac{x^2}{2} + 4\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 4x - 9\frac{x^{\frac{5}{3}}}{\frac{5}{3}}\right]_0^1 = \frac{53}{30}$$

(b) Sketch the region bounded by the curve $y = 2 + \sin x$, $x \in \left[0, \frac{\pi}{2}\right]$ and lines $y = 3$, $x = 0$ and find the volume of the solid generated by revolving this region about the $x$-axis.

$$2 + \sin x = 3, x \in \left[0, \frac{\pi}{2}\right] \implies x = \frac{\pi}{2} \text{ and } -1 \leq \sin x \leq 1 \implies 2 + \sin x \leq 3$$



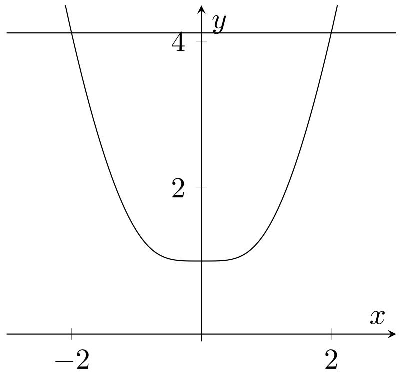

$$\text{Volume} = \pi \int_0^{\frac{\pi}{2}} (3)^2 - (2+\sin x)^2 dx = \pi \int_0^{\frac{\pi}{2}} 9 - 4 - 4\sin x - \sin^2 x dx = \pi \int_0^{\frac{\pi}{2}} 5 - 4\sin x - \frac{1-\cos 2x}{2} dx$$

$$\overset{\text{FTC}}{=} \pi \left[5x + 4\cos x - \frac{1}{2}x + \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}} = \pi\left(\frac{5\pi}{2} - \frac{\pi}{4} - 4\right) = \frac{9\pi^2}{4} - 4\pi$$

(c) Sketch the region bounded by the curve $y^2 - x^4 = 1$, and line $y = \sqrt{17}$ and find the volume of the solid generated by revolving this region about the $x$-axis.

$$\sqrt{17}^2 - x^4 = 1 \implies x = \pm 2 \text{ and } y^2 - 0^4 = 1 \implies y = 1$$

$$\text{Taking the positive half } y = \sqrt{x^4 + 1}$$

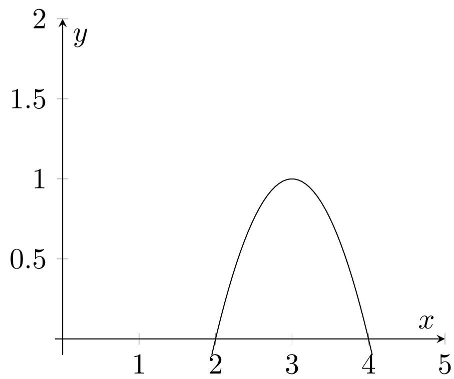



$$\text{Volume} = \pi \int_{-2}^{2} \sqrt{17}^2 - \sqrt{x^4+1}^2 dx = \pi \int_{-2}^{2} 16 - x^4 dx \overset{\text{FTC}}{=} \pi \left[16x - \frac{x^5}{5}\right]_{-2}^{2}$$
$$= \pi \left(\left(16 \cdot 2 - \frac{2^5}{5}\right) - \left(16 \cdot (-2) - \frac{(-2)^5}{5}\right)\right) = \pi \left(64 - \frac{64}{5}\right) = \frac{256\pi}{5}$$

## Exercise 3 (30 points).

(a) Sketch the region bounded by the curve $y = -x^2 + 6x - 8$ and line $y = 0$ and find the volume of the solid generated by revolving this region about the $y$-axis.

$$-x^2 + 6x - 8 = -(x-4)(x-2) = 0 \implies x = 2, x = 4$$

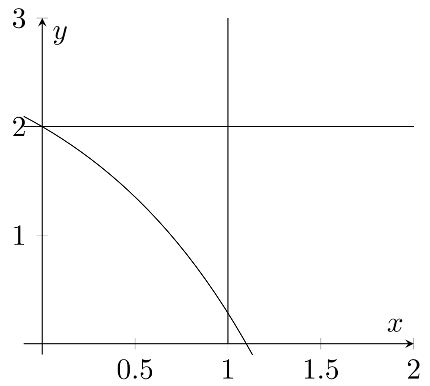



$$\text{Volume} = 2\pi \int_{2}^{4} x(-x^2 + 6x - 8)dx = 2\pi \int_{2}^{4} -x^3 + 6x^2 - 8x dx$$

$$= 2\pi \left[-\frac{x^4}{4} + 6\frac{x^3}{3} - 8\frac{x^2}{2}\right]_{2}^{4} = 2\pi \left(\left(-\frac{4^4}{4} + 2 \cdot 4^3 - 4 \cdot 4^2\right) - \left(-\frac{2^4}{4} + 2 \cdot 2^3 - 4 \cdot 2^2\right)\right) = 8\pi$$

(b) Sketch the region bounded by the curve $y = 3 - e^x$ and lines $y = 2$, $x = 1$ and find the volume of the solid generated by revolving this region about the $y$-axis.

$$2 = 3 - e^x \implies x = 0 \text{ and } 3 - e^1 = 3 - e$$

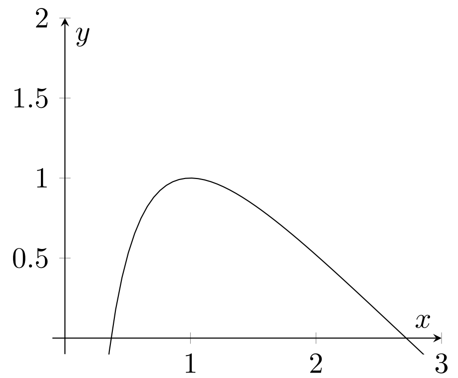



$$\text{Volume} = 2\pi \int_0^1 x(3 - e^x)dx = 2\pi \int_0^1 3x - 2\pi \int_0^1 xd\left(e^x\right) \overset{\text{FTC,IbP}}{=} 2\pi \left[3\frac{x^2}{2} - xe^x\right]_0^1 + 2\pi \int_0^1 e^x dx$$

$$\overset{\text{FTC}}{=} 2\pi \left[3\frac{x^2}{2} - xe^x + e^x\right]_0^1 = 2\pi \left(\left(\frac{3 \cdot 1^2}{2} - 1 \cdot e^1 + e^1\right) - \left(\frac{3 \cdot 0^2}{2} - 0 \cdot e^0 + e^0\right)\right) = \pi$$

(c) Sketch the region bounded by the curve $y = 1 - \ln^2 x$ and line $y = 0$ and find the volume of the solid generated by revolving this region about the $y$-axis.

$$0 = 1 - \ln^2 x \implies x = \frac{1}{e}, x = e, \frac{dy}{dx} = -2\frac{\ln x}{x} = 0 \implies x = 1$$




$$\text{Vol.} = 2\pi \int_{\frac{1}{e}}^e x(1 - \ln^2 x)dx \overset{\text{FTC}}{=} 2\pi \left(\left[\frac{x^2}{2}\right]_{\frac{1}{e}}^e - \int_{\frac{1}{e}}^e \ln^2 x d\left(\frac{x^2}{2}\right)\right) \overset{\text{IbP}}{=} 2\pi \left(\left[\frac{x^2(1 - \ln^2 x)}{2}\right]_{\frac{1}{e}}^e + \int_{\frac{1}{e}}^e x \ln x dx\right)$$

$$\overset{\text{IbP}}{=} 2\pi \left(\left[\frac{x^2(1 - \ln^2 x) + x^2 \ln x}{2}\right]_{\frac{1}{e}}^e - \int_{\frac{1}{e}}^e \frac{x}{2} dx\right) \overset{\text{FTC}}{=} 2\pi \left[\frac{x^2(1 - \ln^2 x) + x^2 \ln x}{2} - \frac{x^2}{4}\right]_{\frac{1}{e}}^e$$

$$2\pi \left(\left(\frac{e^2(1 - \ln^2 e) + e^2 \ln e}{2} - \frac{e^2}{4}\right) - \left(\frac{\frac{1}{e}^2(1 - \ln^2 \frac{1}{e}) + \frac{1}{e}^2 \ln \frac{1}{e}}{2} - \frac{\frac{1}{e}^2}{4}\right)\right) = \frac{\pi\left(e^2 + 3e^{-2}\right)}{2}$$