Digital version

# CS101 Calculus 2 Section C - Homework 5


Mher Saribekyan A09210183


February 23, 2024

## Exercise 1 (20 points).

Find the length of the curve:

a) $y = \ln \sin x, \frac{\pi}{3} \leq x \leq \frac{2\pi}{3}$

$$L = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1 + ((\ln \sin x)')^2} dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{1 + \left(\frac{1}{\sin x} \cos x\right)^2} dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sqrt{\frac{\sin^2 x + \cos^2 x}{\sin^2 x}} dx$$

$$\overset{\sin x \geq 0}{=} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{\sin x} dx = \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{\sin x}{(1 - \cos^2 x)} dx = -\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{(1 - \cos^2 x)} d(\cos x)$$

$$\overset{u=\cos x}{=} -\int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{1}{(1-u^2)} du = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{(1+u)(1-u)} du = \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{(1+u)} + \frac{1}{(1-u)} du$$

$$\overset{\text{FTC}}{=} \frac{1}{2} \left[\ln|1+u| - \ln|1-u|\right]_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} = \frac{1}{2} \left[\ln \left|\frac{1+u}{1-u}\right|\right]_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{1}{2} \left(\ln \left|\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right| - \ln \left|\frac{1-\frac{1}{2}}{1+\frac{1}{2}}\right|\right)$$

$$= \frac{1}{2} \left(\ln(3) - \ln\left(\frac{1}{3}\right)\right) = \frac{1}{2} (\ln 3 + \ln 3) = \ln 3$$

b) $y = \int_0^x \sqrt{\cos 2t} dt, 0 \leq x \leq \frac{\pi}{4}$

$$L = \int_0^{\frac{\pi}{4}} \sqrt{1 + \left(\frac{d}{dx} \int_0^x \sqrt{\cos 2t} dt\right)^2} dx \overset{\text{FTC}}{=} \int_0^{\frac{\pi}{4}} \sqrt{1 + \left(\sqrt{\cos 2x}\right)^2} dx \overset{\cos 2x \geq 0}{=} \int_0^{\frac{\pi}{4}} \sqrt{1 + \cos 2x} dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{2 \cos^2 x} dx \overset{\cos x \geq 0}{=} \sqrt{2} \int_0^{\frac{\pi}{4}} \cos x dx = \sqrt{2} [\sin x]_0^{\frac{\pi}{4}} = \frac{\sqrt{2} \cdot \sqrt{2}}{2} = 1$$

## Exercise 2 (40 points).

Find the area of the surface obtained by rotating the curve about the x-axis

a) $y = e^{-x}, 0 \leq x \leq 1$

$$\begin{aligned}
\text{Area} &= \int_0^1 2\pi e^{-x}\sqrt{1+e^{-2x}}dx = 2\pi\int_0^1 \frac{1}{e^x}\sqrt{1+\frac{1}{e^{2x}}}\frac{e^x}{e^x}dx \overset{u=e^x}{=} 2\pi\int_1^e \frac{1}{u^2}\sqrt{1+\frac{1}{u^2}}du \\
&= -2\pi\int_1^e \sqrt{1+\frac{1}{u^2}}\,d\left(\frac{1}{u}\right) \overset{v=\frac{1}{y}}{=} 2\pi\int_{\frac{1}{e}}^1 \sqrt{1+v^2}dv \overset{v=\tan\varphi}{=} 2\pi\int_{\arctan\left(\frac{1}{e}\right)}^{\frac{\pi}{4}} \frac{1}{\cos^3\varphi}d\varphi \\
&\overset{\text{magic}}{=} 2\pi\left[\frac{1}{2}\frac{\tan\varphi}{\cos\varphi}+\frac{1}{4}\ln\left|\frac{1+\sin\varphi}{1-\sin\varphi}\right|\right]_{\arctan\left(\frac{1}{e}\right)}^{\frac{\pi}{4}}
\end{aligned}$$

$$\overset{\text{FTC}}{=} 2\pi\left(\frac{1}{2}\frac{\tan\frac{\pi}{4}}{\cos\frac{\pi}{4}}+\frac{1}{4}\ln\left|\frac{1+\sin\frac{\pi}{4}}{1-\sin\frac{\pi}{4}}\right| - \frac{1}{2}\frac{\tan\arctan\left(\frac{1}{e}\right)}{\cos\arctan\left(\frac{1}{e}\right)} - \frac{1}{4}\ln\left|\frac{1+\sin\arctan\left(\frac{1}{e}\right)}{1-\sin\arctan\left(\frac{1}{e}\right)}\right|\right) = \text{¯\\\_(ツ)\_/¯}$$

$$\begin{aligned}
\text{The magic} \longrightarrow I &=: \int \frac{1}{\cos^3 x}dx = \int \frac{1}{\cos x}d(\tan x) \overset{\text{IbP}}{=} \frac{\tan x}{\cos x} - \int \tan x\frac{\tan x}{\cos x}dx \\
&= \frac{\tan x}{\cos x} - \int\frac{1}{\cos^3 x}dx + \int\frac{1}{\cos x}dx \overset{\text{magic from Ex.1}}{=} \frac{\tan x}{\cos x} - I + \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| \\
I &= \frac{1}{2}\frac{\tan x}{\cos x} + \frac{1}{4}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + c
\end{aligned}$$

b) $y = \sin\pi x, 0 \leq x \leq 1$

$$\begin{aligned}
\text{Area} &= \int_0^1 2\pi\sin(\pi x)\sqrt{1+(\pi\cos(\pi x))^2}dx \overset{u=\pi x}{=} 2\pi\frac{1}{\pi}\int_0^\pi \sin u\sqrt{1+\pi^2\cos^2(u)}du \\
&= -2\int_0^\pi \sqrt{1+\pi^2\cos^2 u}\,d\cos u \overset{v=\cos u}{=} -2\int_1^{-1}\sqrt{1+\pi^2v^2}dv \overset{\text{even}}{=} 4\pi\int_0^1\sqrt{\frac{1}{\pi^2}+v^2}dv \\
&\overset{v=\frac{1}{\pi}\tan\varphi}{=} 4\int_0^{\arctan\pi}\frac{1}{\cos^3\varphi}d\varphi \overset{\text{magic}}{=} 4\left[\frac{1}{2}\frac{\tan\varphi}{\cos\varphi}+\frac{1}{4}\ln\left|\frac{1+\sin\varphi}{1-\sin\varphi}\right|\right]_0^{\arctan\pi} \\
&= 4\left(\frac{1}{2}\frac{\tan\arctan\pi}{\cos\arctan\pi}+\frac{1}{4}\ln\left|\frac{1+\sin\arctan\pi}{1-\sin\arctan\pi}\right|\right) = \text{¯_(ツ)_/¯}
\end{aligned}$$

c) $x = 1 + 2y^2, 1 \leq y \leq 2$

Take the inverse function $y = \frac{1}{2}\sqrt{x-1}, x \in [3, 9]$

$$\begin{aligned}
\text{Area} &= \int_3^9 2\pi\frac{1}{2}\sqrt{x-1}\sqrt{1+\left(\frac{1}{4\sqrt{x-1}}\right)^2}dx = \pi\int_3^9\sqrt{x-1}\frac{\sqrt{16x-15}}{4\sqrt{(x-1)}}dx \\
&= \frac{\pi}{4}\int_3^9\sqrt{16x-15}dx = \frac{\pi}{4}\left[\frac{2}{48}(16x-15)^{\frac{3}{2}}\right]_3^9 = \frac{\pi}{96}\left(129\sqrt{129}-33\sqrt{33}\right)
\end{aligned}$$

d) $y = \sqrt{36-x^2}, -6 \leq x \leq 6$

$$\begin{aligned}
\text{Area} &= \int_{-6}^6 2\pi\sqrt{36-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{36-x^2}}\right)^2}dx = \int_{-6}^6 2\pi\sqrt{36-x^2}\frac{\sqrt{36}}{\sqrt{36-x^2}}dx \\
&= 2\pi\int_{-6}^6 6dx = 2\pi\cdot 72 = 144\pi
\end{aligned}$$

## Exercise 3 (20 points).

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the following parametric equations:

a) $x = e^t \sin t$, $y = 2t^3 \ln t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dy}{dx}} = \frac{6t^2 \ln t + 2t^2}{e^t \sin t + e^t \cos t} = \frac{2t^2(3\ln t + 1)}{e^t(\sin x + \cos x)}$$

$$\frac{d^2y}{dx^2} = \frac{d\frac{dy}{dx}}{dx} = \frac{\frac{d\frac{dy}{dx}}{dt}}{\frac{dx}{dt}} = \frac{\frac{e^t(\sin t+\cos t)(12t\ln t+2t)+2t^2(3\ln t+1)\left(e^t(\sin t+\cos t)+e^t(\cos t-\sin t)\right)}{(e^t(\sin t+\cos t))^2}}{e^t(\sin t + \cos t)}$$

$$= \frac{e^t(\sin t + \cos t)\,(12t\ln t + 2t) + 2t^2(3\ln t + 1)\left(2e^t \cos t\right)}{\left(e^t(\sin t + \cos t)\right)^3}$$

b) $x = \frac{1}{t+1}$, $y = \frac{t^2}{t-1}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{2t}{t-1} - \frac{t^2}{(t-1)^2}}{-\frac{1}{(t+1)^2}} = -(t+1)^2\left(\frac{2t}{t-1} - \frac{t^2}{(t-1)^2}\right)$$

$$\frac{d^2y}{dx^2} = \frac{-2(t+1)\left(\frac{t^2-2t}{(t-1)^2}\right) - (t+1)^2\frac{(t-1)^2(2t-2)-2(t+1)(t^2-2t)}{(t-1)^4}}{-\frac{1}{(t+1)^2}}$$

$$= 2(t+1)^3\left(\frac{t^2-2t}{(t-1)^2}\right) + (t+1)^4\frac{(t-1)^2(2t-2)-2(t+1)(t^2-2t)}{(t-1)^4}$$

## Exercise 4 (20 points).

Find the equation of the tangent line to the following curves:

a) $x = 1 + 3\sin t$, $y = 2 - 5\cos t$ at $t = \frac{\pi}{6}$

$$\frac{dy}{dx} = \frac{5\sin t}{3\cos t} = \frac{5}{3}\tan t \implies \left.\frac{dy}{dx}\right|_{t=\frac{\pi}{6}} = \frac{5\sqrt{3}}{9}$$

$$\text{Take } t_0 = \frac{\pi}{6} \implies x_0 = \frac{5}{2}, y_0 = \frac{4-5\sqrt{3}}{2} \implies y = \frac{5\sqrt{3}}{9}\left(x - \frac{5}{2}\right) + \frac{4-5\sqrt{3}}{2} = \frac{5\sqrt{3}}{9}x + \frac{18-35\sqrt{3}}{9}$$

b) $x = (t-1)^3 \ln t + e^{2t-2}$, $y = \sin^2(\pi t) + 5\ln(2t-1)$ at $t = 1$

$$\frac{dy}{dx} = \frac{2\sin(\pi t)\cos(\pi t)\pi + \frac{10}{2t-1}}{3(t-1)^2\ln t + \frac{(t-1)^3}{t} + 2e^{2t-2}} \implies \left.\frac{dy}{dx}\right|_{t=1} = 5$$

$$\text{Take } t_0 = 1 \implies x_0 = 1, y_0 = 0 \implies y = 5x - 5$$