Digital version

CS101 Calculus 2 Section C - Homework 6

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Exercise 1 (40 points).

Find the length of the following parametric curves:

a) cardioid: $x = a(2\cos t - \cos 2t)$, $y = a(2\sin t - \sin 2t)$, $t \in [0, 2\pi]$ and a is a fixed positive number.

$$L = \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2}dt = \int_0^{2\pi} \sqrt{(a(2\sin 2t - 2\sin t))^2 + (a(2\cos t - 2\cos 2t))^2}dt$$

$$= 2a \int_0^{2\pi} \sqrt{\sin^2 2t - 2\sin 2t \sin t + \sin^2 t + \cos^2 t - 2\cos t \cos 2t + \cos^2 2t}dt$$

$$= 2a \int_0^{2\pi} \sqrt{2 - 2(\sin 2t \sin t + \cos t \cos 2t)}dt = 2\sqrt{2}a \int_0^{2\pi} \sqrt{1 - (2\sin^2 t \cos t + \cos t - 2\cos t \sin^2 t)}dt$$

$$= 2\sqrt{2}a \int_0^{2\pi} \sqrt{1 - \cos t}dt = 2\sqrt{2}a \int_0^{2\pi} \sqrt{2\sin^2\left(\frac{t}{2}\right)}dt = 4a \int_0^{2\pi} \sin\left(\frac{t}{2}\right)dt$$

$$= 4a \left[-2\cos\left(\frac{t}{2}\right)\right]_0^{2\pi} = 8a\left(1 - (-1)\right) = 16a$$

b) $x = \frac{1}{6}(4t+8)^{\frac{3}{2}}$, $y = 2t + t^2$, $t \in [0, 2]$.

$$L = \int_0^2 \sqrt{(x'(t)^2 + y'(t)^2}dt = \int_0^2 \sqrt{\left(\sqrt{4t+8}\right)^2 + (2+2t)^2}dt = \int_0^2 \sqrt{4t + 8 + 4 + 8t + 4t^2}dt$$

$$= 2\int_0^2 \sqrt{t^2 + 3t + 3}dt = 2\int_0^2 \sqrt{\left(t + \frac{3}{2}\right)^2 + \frac{3}{4}}dt \overset{t + \frac{3}{2} = \frac{\sqrt{3}}{2}\tan\varphi}{=} \frac{3}{2}\int_{\frac{\pi}{3}}^{\arctan\left(\frac{7\sqrt{3}}{3}\right)} \frac{1}{\cos^3\varphi}d\varphi$$

$$\overset{\text{magic}}{=} \left[\frac{1}{2}\frac{\tan\varphi}{\cos\varphi} + \frac{1}{4}\ln\left|\frac{1+\sin\varphi}{1-\sin\varphi}\right|\right]_{\frac{\pi}{3}}^{\arctan\left(\frac{7\sqrt{3}}{3}\right)}$$

$$= \left(\frac{1}{2}\frac{\frac{7\sqrt{3}}{3}}{\frac{\sqrt{3}}{2\sqrt{13}}} + \frac{1}{4}\ln\left|\frac{1+\frac{7}{2\sqrt{13}}}{1-\frac{7}{2\sqrt{13}}}\right|\right) - \left(\frac{1}{2}\frac{\sqrt{3}}{\frac{1}{2}} + \frac{1}{4}\ln\left|\frac{1+\frac{\sqrt{3}}{2}}{1-\frac{\sqrt{3}}{2}}\right|\right)$$

The magic used $\longrightarrow I =: \int \frac{1}{\cos^3 x}dx = \int \frac{1}{\cos x}d(\tan x) \overset{\text{IbP}}{=} \frac{\tan x}{\cos x} - \int \tan x \frac{\tan x}{\cos x}dx$

$$= \frac{\tan x}{\cos x} - \int \frac{1}{\cos^3 x} dx + \int \frac{1}{\cos x} dx \overset{\text{other magic}}{=} \frac{\tan x}{\cos x} - I + \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right|$$

$$I = \frac{1}{2} \frac{\tan x}{\cos x} + \frac{1}{4} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right| + c$$

$$\text{The other magic used} \longrightarrow \int \frac{1}{\cos x} dx = \int \frac{\cos x}{(1 - \sin^2 x)} dx = \int \frac{1}{(1 - \sin^2 x)} d(\sin x)$$

$$\overset{u=\sin x}{=} \int \frac{1}{1 - u^2} du = \int \frac{1}{(1+u)(1-u)} du = \frac{1}{2} \int \frac{1}{(1+u)} + \frac{1}{(1-u)} du$$

$$\overset{\text{FTC}}{=} \frac{1}{2} [\ln|1+u| - \ln|1-u|] = \frac{1}{2} \left(\ln \left| \frac{1 + \sin x}{1 - \sin x} \right| \right) + c$$

c) $x = 2t^6$, $y = 3t^4$, $t \in [0, 1]$.

$$L = \int_0^1 \sqrt{x'(t)^2 + y'(t)^2} dt = \int_0^1 \sqrt{(12t^5)^2 + (12t^3)^2} dt = 12 \int_0^1 \sqrt{t^{10} + t^6} dt = 12 \int_0^1 t^3 \sqrt{t^4 + 1} dt$$

$$= 12 \int_0^1 \frac{1}{4} \sqrt{t^4 + 1} d(t^4 + 1) \overset{u=t^4+1}{=} 3 \int_1^2 \sqrt{u} du = 3 \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right]_1^2 = 2 \left[u^{\frac{3}{2}} \right]_1^2 = 4\sqrt{2} - 2$$

d) $x = e^{-t} \cos(t)$, $y = e^{-t} \sin(t)$, $t \in [0, 1]$.

$$L = \int_0^1 \sqrt{x'(t)^2 + y'(t)^2} dt = \int_0^1 \sqrt{(-e^{-t} \sin(t) - e^{-t} \cos(t))^2 + (e^{-t} \cos(t) - e^{-t} \sin(t))^2} dt$$

$$= \int_0^1 e^{-t} \sqrt{\sin^2(t) + 2\sin(t)\cos(t) + \cos^2(t) + \cos^2(t) - 2\cos(t)\sin(t) + \sin^2(t)} dt$$

$$= \sqrt{2}[-e^{-t}]_0^1 = \sqrt{2}\left((-e^{-1}) - (-e^{-0})\right) = \sqrt{2}\left(-\frac{1}{e} + 1\right) = \frac{\sqrt{2}(e-1)}{e}$$

Exercise 2 (40 points).

Find the area enclosed by the following curves:

a) $x = 3t - t^2$, $y = e^t$ and the y-axis.

$$x = 0 \implies 3t - t^2 = 0 \implies t(3-t) = 2 \implies \begin{cases} t = 0 \\ t = 3 \end{cases}$$

$$y' = e^t \geq 0 \text{ and } x = 3t - t^2 \geq 0, t \in [0, 3]$$

$$\text{Area } = \int_0^3 x(t) dy(t) = \int_0^3 x(t) y'(t) dt = \int_0^3 (3t - t^2) e^t dt = \int_0^3 (3t - t^2) d(e^t)$$

$$\overset{\text{IbP}}{=} \left[(3t - t^2)e^t\right]_0^3 - \int_0^e (3 - 2t) e^t dt = \left[(3t - t^2)e^t\right]_0^3 - \int_0^e (3 - 2t) d(e^t)$$

$$\overset{\text{IbP}}{=} \left[(3t - t^2)e^t - (3 - 2t)e^t\right]_0^3 + \int_0^3 (-2) e^t dt = \left[(3t - t^2)e^t - (3 - 2t)e^t - 2e^t\right]_0^3$$

$$= \left(3e^3 - 2e^3\right) - \left(-3e^0 - 2e^0\right) = e^3 + 5$$

b) $x = 2t\sqrt{t+4}$, $y = t^2 + 3t$ and the x-axis.

$$y = 0 \implies t^2 + 3t = 0 \implies t(t+3) = 0 \implies \begin{cases} t = -3 \\ t = 0 \end{cases} \implies y(t) \leq 0, t \in [-3, 0]$$

$$x' = \frac{t}{\sqrt{t+4}} + 2\sqrt{t+4} = \frac{3t+8}{\sqrt{t+4}} \implies \begin{cases} x' \leq 0, x \in [-3, -\frac{8}{3}] \\ x' \geq 0, [-\frac{8}{3}, 0] \end{cases}$$

$$\text{Area } = -\int_{-3}^{0} y(t)x'(t)dt = \int_{0}^{-3} y(t)x'(t)dt \overset{*}{=} \left[\frac{\sqrt{t+4}\left(90t^3 + 282t^2 + 176t - 1408\right)}{105}\right]_0^{-3}$$

$$= \left(\frac{\sqrt{1}\left(90(-3)^3 + 282(-3)^2 + 176(-3) - 1408\right)}{105}\right) - \left(\frac{\sqrt{4}\left(90 \cdot 0^3 + 282 \cdot 0^2 + 176 \cdot 0 - 1408\right)}{105}\right) = \frac{988}{105}$$

$$(*) \int (x^2+3x)\frac{3x+8}{\sqrt{x+4}}dx = \int \frac{(x(x+3+1-1))(3x+8+4-4)}{\sqrt{x+4}}dx \overset{u=x+4}{=} \int \frac{(u-4)(u-1)(3u-4)}{\sqrt{u}}du$$

$$= \int \frac{3u^3 - 4u^2 - 3u^2 + 4u - 12u^2 + 16u + 12u - 16}{\sqrt{u}}du = \int \frac{3u^3 - 19u^2 + 32u - 16}{\sqrt{u}}du$$

$$= \int 3u^{\frac{5}{2}} - 19u^{\frac{3}{2}} + 32u^{\frac{1}{2}} - 16u^{-\frac{1}{2}}du = \left(\frac{6(x+4)^{\frac{7}{2}}}{7} - \frac{38(x+4)^{\frac{5}{2}}}{5} + \frac{64(x+4)^{\frac{3}{2}}}{3} - 32(x+4)^{\frac{1}{2}}\right)$$

$$= \sqrt{x+4}\left(\frac{6(x+4)^3}{7} - \frac{38(x+4)^2}{5} + \frac{64(x+4)}{3} - 32\right)$$

$$= \frac{\sqrt{x+4}\left(90(x^3 + 12x^2 + 48x + 64) - 798(x^2 + 8x + 16)^2 + 2240(x+4) - 3360\right)}{105}$$

$$= \frac{\sqrt{x+4}\left(90x^3 + 282x^2 + 176x - 1408\right)}{105} + c$$

c) $x = \arcsin t$, $y = t^3$, the x-axis and $x = \frac{\pi}{6}$.

$$y = 0 \implies t = 0 \text{ and } x = \frac{\pi}{6} \implies t = \frac{1}{2}$$

$$x' = \frac{1}{\sqrt{1-t^2}} \geq 0, y \geq 0, t \in \left[0, \frac{1}{2}\right]$$

$$\text{Area } = \int_0^{\frac{1}{2}} y(t)dx(t) = \int_0^{\frac{1}{2}} \frac{t^3}{\sqrt{1-t^2}}dt = \frac{1}{2}\int_0^{\frac{1}{2}} \frac{(1-t^2)-1}{\sqrt{1-t^2}}d(1-t^2) \overset{u=1-t^2}{=} \frac{1}{2}\int_1^{\frac{3}{4}} \frac{u-1}{\sqrt{u}}du$$

$$= \frac{1}{2}\int_{\frac{3}{4}}^{1} u^{-\frac{1}{2}} - u^{\frac{1}{2}}du = \frac{1}{2}\left[2u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right]_{\frac{3}{4}}^{1} = \frac{1}{2}\left(\left(2 - \frac{2}{3}\right) - \left(\sqrt{3} - \frac{\sqrt{3}}{4}\right)\right) = \frac{2}{3} - \frac{3\sqrt{3}}{8} = \frac{16 - 9\sqrt{3}}{24}$$

d) cardoid: $x = a(2\cos t - \cos 2t)$, $y = a(2\sin t - \sin 2t)$, $t \in [0, 2\pi]$.
As cardoid is symmetric on the x-axis, we can find the area from $t = 0$ to $t = \pi$ and multiply by 2.

$$x'(t) = a(2\sin 2t - 2\sin t) \implies \begin{cases} x'(t) \geq 0, t \in \left[0, \frac{\pi}{3}\right] \\ x'(t) \leq 0, t \in \left[\frac{\pi}{3}, \pi\right] \end{cases}$$

$$y=0 \implies 2a(\sin t-\sin t\cos t)=0 \implies \sin t(1-\cos t)=0 \implies \begin{cases} t=0 \\ t=\pi \end{cases} \implies y(t)\geq 0, t\in[0,\pi]$$

$$\begin{aligned}
\text{Area } &= -2\int_0^\pi y(t)dx(t) = -2\int_0^\pi a(2\sin t-\sin 2t)a(2\sin 2t-2\sin t)dt \\
&= -2a^2\int_0^\pi 4\sin t\sin 2t-4\sin^2 t-2\sin^2 2t+2\sin 2t\sin t dt \\
&= -2a^2\left(12\int_0^\pi \sin^2 t\cos t dt-2\int_0^\pi 1-\cos 2t dt-\int_0^\pi 1-\cos 4t dt\right) \\
&= -2a^2\left[4\sin^3 t+\frac{\sin(4t)}{4}+\sin(2t)-3t\right]_0^\pi
\end{aligned}$$

$$= -2a^2\left(\left(4\sin^3\pi+\frac{\sin(4\pi)}{4}+\sin(2\pi)-3\pi\right)-\left(4\sin^3 0+\frac{\sin(4\cdot 0)}{4}+\sin(2\cdot 0)-3\cdot 0\right)\right)=6\pi a^2$$

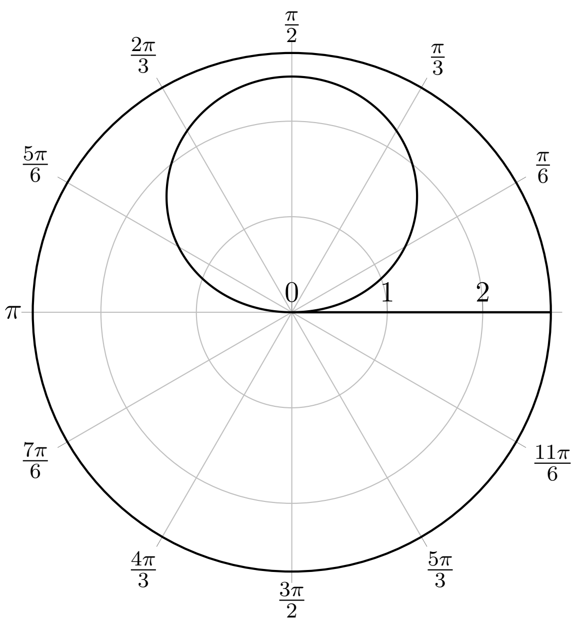
Exercise 3 (20 points).

Sketch the polar curve (please note that in case the limits of the variable φ are not specified, you need to identify all that values of φ such that $r=r(\varphi)\geq 0$):

a) $r=\pi\varphi-\varphi^2$

$$r=\pi\varphi-\varphi^2 \longrightarrow \pi\varphi-\varphi^2=0 \implies \varphi(\pi-\varphi)=0 \implies \begin{cases} \varphi=0 \\ \varphi=\pi \end{cases} \overset{r\geq 0}{\implies} \varphi\in[0,\pi]$$

$$r'(\varphi)=\pi-2\varphi \implies \begin{cases} r\uparrow, \varphi\in\left[0,\frac{\pi}{2}\right] \\ r\downarrow, \varphi\in\left[\frac{\pi}{2},\pi\right] \end{cases}, r_{min}=r(0)=0, r_{max}=\pi\frac{\pi}{2}-\left(\frac{\pi}{2}\right)^2=\frac{\pi^2}{4}$$



b) $r = \sin 4\varphi + 1$

$$-1 \le \sin 4\varphi \le 1 \implies r \ge 0, \sin 4\varphi \text{ is } \frac{\pi}{2} \text{ periodic} \implies \text{also } 2\pi \text{ periodic} \implies \varphi \in [0, 2\pi]$$

$$\text{Take period } \frac{\pi}{2} \text{ (for sketching)}, \ r'(\varphi) = 4\cos 4\varphi \implies \begin{cases} r\uparrow, \varphi \in \left[0, \frac{\pi}{8}\right], \left[\frac{3\pi}{8}, \frac{\pi}{2}\right] \\ r\downarrow, \varphi \in \left[\frac{\pi}{8}, \frac{3\pi}{8}\right] \end{cases}$$

$$r\left(0\right) = r\left(\frac{\pi}{4}\right) = r\left(\frac{\pi}{2}\right) = 1, r\left(\frac{\pi}{8}\right) = 2, r\left(\frac{3\pi}{8}\right) = 0, \text{ repeat for all four quadrants} \left(+\frac{\pi}{2}\right)$$