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CS101 Calculus 2 Section C - Homework 7

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Exercise 1 (10 points).

Find the area of the following polar (curvilinear) sector:

$$r = 5 + \cos\varphi, \varphi \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right].$$

$$\text{Area } = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (5+\cos\varphi)^2 \ \mathrm{d}\varphi = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 25 + 10\cos\varphi + \cos^2\varphi \ \mathrm{d}\varphi$$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 25 + 10\cos\varphi + \frac{1+\cos 2\varphi}{2} \ \mathrm{d}\varphi = \frac{1}{2}\left[25\varphi + 10\sin\varphi + \frac{1}{2}\varphi + \frac{\sin 2\varphi}{4}\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$= \frac{1}{2}\left(\left(\frac{51}{2}\frac{3\pi}{4} + 10\sin\frac{3\pi}{4} + \frac{\sin 2\frac{3\pi}{4}}{4}\right) - \left(\frac{51}{2}\frac{\pi}{4} + 10\sin\frac{\pi}{4} + \frac{\sin 2\frac{\pi}{4}}{4}\right)\right) = \frac{51\pi - 2}{8}$$

Exercise 2 (20 points).

Find the area of the region that lies inside the first curve and outside the second curve.

a) $r = 2\cos\varphi$; $r = 1$,

$$2\cos\varphi = 1 \implies \cos\varphi = \frac{1}{2} \implies \varphi = \left\{-\frac{\pi}{3}, \frac{\pi}{3}\right\} \implies 2\cos\varphi \geq 1, \varphi \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$$

$$\text{Area } = \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (2\cos\varphi)^2 - (1)^2 \ \mathrm{d}\varphi = \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 2 + 2\cos 2\varphi - 1 \ \mathrm{d}\varphi \overset{\text{even}}{=} [\varphi + \sin 2\varphi]_0^{\frac{\pi}{3}}$$

$$= \left(\frac{\pi}{3} + \sin\left(\frac{2\pi}{3}\right)\right) = \frac{\pi}{3} + \frac{\sqrt{3}}{2} = \frac{2\pi + 3\sqrt{3}}{6}$$

b) $r = 2 + \sin\varphi$; $r = 3\sin\varphi$.

$$\sin\varphi \leq 1 \implies 2 + \sin\varphi \geq 3\sin\varphi, \ \varphi \in [0, \pi]$$

$$\text{Area } = \frac{1}{2}\int_0^{2\pi} (2+\sin\varphi)^2 \,\mathrm{d}\varphi - \frac{1}{2}\int_0^{\pi} (3\sin\varphi)^2 \,\mathrm{d}\varphi = \frac{1}{2}\int_0^{2\pi} 4 + 4\sin\varphi + \sin^2\varphi \,\mathrm{d}\varphi - \frac{1}{2}\int_0^{\pi} 9\sin^2\varphi \,\mathrm{d}\varphi$$

$$= \frac{1}{2}\left[4\varphi - 4\cos\varphi + \frac{1}{2}\varphi - \frac{\sin 2\varphi}{4}\right]_0^{2\pi} - \frac{9}{2}\left[\frac{1}{2}\varphi - \frac{\sin 2\varphi}{4}\right]_0^{\pi} = \frac{9}{2}\pi - \frac{9}{4}\pi = \frac{9}{4}\pi$$

Exercise 3 (20 points).

Find the length of the polar curve.

a) $r = e^{3\varphi}$, $\varphi \in [0, \pi]$ (spiral),

$$\text{Length } = \int_0^{\pi} \sqrt{(e^{3\varphi})^2 + (3e^{3\varphi})^2}\ \mathrm{d}\varphi = \int_0^{\pi} \sqrt{10e^{6\varphi}}\ \mathrm{d}\varphi = \frac{\sqrt{10}}{3}\left[e^{3\varphi}\right]_0^{\pi} = \frac{\sqrt{10}}{3}(e^{3\pi} - 1)$$

b) $r = \frac{5}{\cos\varphi}$, $\varphi \in \left[0, \frac{\pi}{4}\right]$.

$$\begin{aligned}\text{Length } &= \int_0^{\frac{\pi}{4}} \sqrt{\left(\frac{5}{\cos\varphi}\right)^2 + \left(\frac{5\sin\varphi}{\cos^2\varphi}\right)^2}\ \mathrm{d}\varphi = \int_0^{\frac{\pi}{4}} \sqrt{\frac{25}{\cos^2\varphi} + \frac{25\sin^2\varphi}{\cos^4\varphi}}\ \mathrm{d}\varphi \\ &= \int_0^{\frac{\pi}{4}} \sqrt{\frac{25\cos^2\varphi + 25\sin^2\varphi}{\cos^4\varphi}}\ \mathrm{d}\varphi = \int_0^{\frac{\pi}{4}} \frac{5}{\cos^2\varphi}\ \mathrm{d}\varphi = 5\left[\tan\varphi\right]_0^{\frac{\pi}{4}} = 5\end{aligned}$$

Exercise 4 (50 points).

Calculate the following improper integrals or prove that they diverge:

a) $\int_3^{+\infty} \frac{1}{x^2+4x-5}\ \mathrm{d}x$,

$$\begin{aligned}&\int_3^{+\infty} \frac{1}{x^2+4x-5}\ \mathrm{d}x = \int_3^{+\infty} \frac{1}{(x-1)(x+5)}\ \mathrm{d}x = \frac{1}{6}\int_3^{+\infty} \frac{1}{x-1}\ \mathrm{d}x - \frac{1}{6}\int_3^{+\infty} \frac{1}{x+5}\ \mathrm{d}x \\ &= \frac{1}{6}\left[\ln\left(\frac{x-1}{x+5}\right)\right]_3^{+\infty} = \frac{1}{6}\left(\lim_{c\to+\infty} \ln\left(\frac{c-1}{c+5}\right) - \ln\left(\frac{3-1}{3+5}\right)\right) = -\frac{1}{6}\cdot\ln\left(\frac{2}{8}\right) = \frac{1}{3}\ln 2\end{aligned}$$

b) $\int_0^{+\infty} \cos\left(\frac{\pi x}{3}\right)\ \mathrm{d}x$,

$$\int_0^{+\infty} \cos\left(\frac{\pi x}{3}\right)\ \mathrm{d}x = \left[\frac{3}{\pi}\sin\left(\frac{\pi x}{3}\right)\right]_0^{+\infty} = \lim_{c\to+\infty} \frac{3}{\pi}\sin\left(\frac{\pi c}{3}\right) - 0 \implies \text{ does not exist } \implies \text{ diverges}$$

c) $\int_0^{+\infty} \frac{1}{(4+\arctan(x))^2(x^2+1)}\ \mathrm{d}x$,

$$\begin{aligned}&\int_0^{+\infty} \frac{1}{(4+\arctan(x))^2(x^2+1)}\ \mathrm{d}x = \int_0^{+\infty} \frac{1}{(4+\arctan(x))^2}\ \mathrm{d}(4+\arctan(x)) \\ &\overset{u=4+\arctan(x)}{=} \int_4^{4+\frac{\pi}{2}} \frac{1}{u^2}du = \left[-\frac{1}{u}\right]_4^{4+\frac{\pi}{2}} = -\frac{1}{4+\frac{\pi}{2}} + \frac{1}{4} = \frac{-4+4+\frac{\pi}{2}}{16+2\pi} = \frac{\pi}{32+4\pi}\end{aligned}$$

d) $\int_{-\infty}^0 \frac{x}{x^4+1}\ \mathrm{d}x$,

$$\begin{aligned}&\int_{-\infty}^0 \frac{x}{x^4+1}\ \mathrm{d}x = \frac{1}{2}\int_{-\infty}^0 \frac{\mathrm{d}x^2}{(x^2)^2+1} \overset{u=x^2}{=} \frac{1}{2}\int_{\infty}^0 \frac{\mathrm{d}u}{u^2+1} \overset{u=\tan\varphi}{=} -\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\tan\varphi}{\tan^2\varphi+1} \\ &= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\frac{1}{\cos^2\varphi}}{\frac{1}{\cos^2\varphi}}\ \mathrm{d}\varphi \overset{\text{FTC}}{=} -\frac{1}{2}\left[\varphi\right]_0^{\frac{\pi}{2}} = -\frac{\pi}{4}\end{aligned}$$

e) $\int_{-\infty}^{+\infty} |x|e^{-x^2}\ \mathrm{d}x$.

$$\int_{-\infty}^{+\infty} |x|e^{-x^2}\ \mathrm{d}x \overset{\text{even}}{=} 2\int_0^{+\infty} xe^{-x^2}\ \mathrm{d}x \overset{u=x^2}{=} \int_0^{+\infty} e^{-u}\ \mathrm{d}u = \left[-e^{-u}\right]_0^{+\infty} = \lim_{c\to+\infty} -\frac{1}{e^u} + e^{-0} = 1$$