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CS101 Calculus 2 Section C - Homework 8

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Exercise 1 (60 points).

Determine the convergence of the following improper integrals of non-negative integrands:

a) $\int_3^{+\infty} \frac{1+2\sqrt[4]{x}}{x+3\sqrt[3]{x}-2}\,\mathrm{d}x$,

$$\lim_{x\to+\infty} \frac{\frac{1+2\sqrt[4]{x}}{x+3\sqrt[3]{x}-2}}{\frac{2}{x^{\frac{3}{4}}}} = \lim_{x\to+\infty} \frac{\frac{1}{2}x^{\frac{3}{4}} + x^{\frac{3}{4}}\sqrt[4]{x}}{x+3\sqrt[3]{x}-2} = \lim_{x\to+\infty} \frac{\frac{1}{x^{\frac{1}{4}}}+1}{1+\frac{3}{x^{\frac{2}{3}}}-\frac{2}{x}} = 1$$

$$\int_3^{+\infty} \frac{2}{x^{\frac{3}{4}}}\,\mathrm{d}x \xrightarrow{\frac{3}{4}\le 1} \text{diverges} \overset{\text{LCT}}{\Longrightarrow} \int_3^{+\infty} \frac{1+2\sqrt[4]{x}}{x+3\sqrt[3]{x}-2}\,\mathrm{d}x \longrightarrow \text{diverges}$$

b) $\int_{-\infty}^{-1} \frac{2x^2+3}{\sqrt[3]{2x^2-x^{11}}}\,\mathrm{d}x$,

$$\lim_{x\to-\infty} \frac{\frac{2x^2+3}{\sqrt[3]{2x^2-x^{11}}}}{-\frac{2}{x^{\frac{5}{3}}}} = \lim_{x\to-\infty} \frac{-x^{\frac{11}{3}}-\frac{3}{2}x^{\frac{5}{3}}}{\sqrt[3]{2x^2-x^{11}}} = \lim_{x\to-\infty} \frac{-1-\frac{3}{2}\frac{1}{x^2}}{\sqrt[3]{\frac{1}{x^9}-1}} = 1$$

$$\int_{-\infty}^{-1} -\frac{2}{x^{\frac{5}{3}}}\,\mathrm{d}x \xrightarrow{\frac{5}{3}>1} \text{converges} \overset{\text{LCT}}{\Longrightarrow} \int_{-\infty}^{-1} \frac{2x^2+3}{\sqrt[3]{2x^2-x^{11}}}\,\mathrm{d}x \longrightarrow \text{converges}$$

c) $\int_1^{+\infty} \frac{3-2\cos(3^x)}{(4\sqrt[3]{x}-1)^3}\,\mathrm{d}x$,

$$\frac{3-2\cos(3^x)}{(4\sqrt[3]{x}-1)^3} \ge \frac{1}{(4\sqrt[3]{x})^3} = \frac{1}{64x}$$

$$\int_1^{+\infty} \frac{1}{64x}\,\mathrm{d}x \xrightarrow{1\le 1} \text{diverges} \overset{\text{DCT}}{\Longrightarrow} \int_1^{+\infty} \frac{3-2\cos(3^x)}{(4\sqrt[3]{x}-1)^3}\,\mathrm{d}x \longrightarrow \text{diverges}$$

d) $\int_1^{+\infty} \frac{3+x-2\cos(3^x)}{4\sqrt[3]{x^7}-x^2+4}\,\mathrm{d}x$,

$$\frac{3+x-2\cos(3^x)}{4\sqrt[3]{x^7}-x^2+4} \le \frac{x+5}{4\sqrt[3]{x^7}-x^2+4}$$

$$\lim_{x\to+\infty} \frac{\frac{x+5}{4\sqrt[3]{x^7}-x^2+4}}{\frac{4}{x^{\frac{4}{3}}}} = \lim_{x\to+\infty} \frac{4x^{\frac{7}{3}}+20x^{\frac{4}{3}}}{4x^{\frac{7}{3}}-x^2+4} = \lim_{x\to+\infty} \frac{4+20\frac{1}{x}}{4-\frac{1}{x^{\frac{1}{3}}}} = 1$$

$$\int_1^{+\infty} \frac{4}{x^{\frac{4}{3}}}\ \mathrm{d}x \overset{\frac{4}{3}>1}{\longrightarrow} \text{converges} \overset{\text{LCT}}{\Longrightarrow} \int_1^{+\infty} \frac{x+5}{4\sqrt[3]{x^7}-x^2+4}\ \mathrm{d}x \longrightarrow \text{converges}$$

$$\overset{\text{DCT}}{\Longrightarrow} \int_1^{+\infty} \frac{3+x-2\cos(3^x)}{4\sqrt[3]{x^7}-x^2+4}\ \mathrm{d}x \longrightarrow \text{converges}$$

e) $\int_1^{+\infty} \left(1-\cos\frac{4}{x}\right)\ \mathrm{d}x$,

$$\int_1^{+\infty} \left(1-\cos\frac{4}{x}\right)\ \mathrm{d}x = \int_1^{+\infty} 2\sin^2\frac{2}{x}\ \mathrm{d}x$$

$$\lim_{x\to+\infty}\left(\frac{\sin\frac{2}{x}}{\frac{2}{x}}\right)^2 = 1 \text{ and } \int_1^{+\infty} \frac{4}{x^2} \overset{2>1}{\to} \text{converges} \Longrightarrow \int_1^{+\infty} \left(1-\cos\frac{4}{x}\right)\ \mathrm{d}x \to \text{converges}$$

f) $\int_1^{+\infty} \frac{\ln\left(1+x^3\right)}{x+1}\ \mathrm{d}x$,

$$\frac{\ln\left(1+x^3\right)}{x+1} \geq \frac{\ln(1+1)}{x+1} = \frac{\ln(2)}{x+1}$$

$$\int_1^{+\infty} \frac{\ln(2)}{x+1}\ \mathrm{d}x \to \text{converges} \overset{\text{DCT}}{\Longrightarrow} \int_1^{+\infty} \frac{\ln\left(1+x^3\right)}{x+1}\ \mathrm{d}x \to \text{converges}$$

g) $\int_{-\infty}^{+\infty} \frac{1}{2(0.3)^x+4\cdot 3^x+2}\ \mathrm{d}x$,

$$\int_{-\infty}^{+\infty} \frac{1}{2(0.3)^x+4\cdot 3^x+2}\ \mathrm{d}x = \int_{-\infty}^{0} \frac{1}{2(0.3)^x+4\cdot 3^x+2}\ \mathrm{d}x + \int_0^{+\infty} \frac{1}{2(0.3)^x+4\cdot 3^x+2}\ \mathrm{d}x$$

$$\lim_{x\to-\infty} \frac{\frac{1}{2(0.3)^x+4\cdot 3^x+2}}{\frac{1}{2\cdot 0.3^x}} = \frac{2\cdot 0.3^x}{0.3^x\left((2+4\cdot 10^x\frac{2}{0.3^x}\right)} = 1$$

$$\int_{-\infty}^{0} \frac{1}{2\cdot 0.3^x}\ \mathrm{d}x \overset{0.3<1}{\to} \text{converges} \overset{\text{LCT}}{\Longrightarrow} \int_{-\infty}^{0} \frac{1}{2(0.3)^x+4\cdot 3^x+2}\ \mathrm{d}x{\to}\text{converges}$$

$$\lim_{x\to+\infty} \frac{\frac{1}{2(0.3)^x+4\cdot 3^x+2}}{\frac{1}{4\cdot 3^x}} = \frac{4\cdot 3^x}{3^x\left((2(0.1)^x+4+\frac{2}{3^x}\right)} = 1$$

$$\int_0^{+\infty} \frac{1}{4\cdot 3^x}\ \mathrm{d}x \overset{3>1}{\to} \text{converges} \overset{\text{LCT}}{\Longrightarrow} \int_0^{+\infty} \frac{1}{2(0.3)^x+4\cdot 3^x+2}\ \mathrm{d}x{\to}\text{converges}$$

$$\therefore \int_{-\infty}^{+\infty} \frac{1}{2(0.3)^x+4\cdot 3^x+2}\ \mathrm{d}x{\longrightarrow}\text{converges}$$

h) $\int_1^{+\infty} \sqrt{2x^3+1}-\sqrt{2x^3-1}\ \mathrm{d}x$,

$$\sqrt{2x^3+1}-\sqrt{2x^3-1} = \frac{\sqrt{2x^3+1}-\sqrt{2x^3-1}}{\sqrt{2x^3+1}+\sqrt{2x^3-1}}\cdot(\sqrt{2x^3+1}+\sqrt{2x^3-1})$$

$$= \frac{2x^3+1-2x^3+1}{\sqrt{2x^3+1}+\sqrt{2x^3-1})} = \frac{2}{\sqrt{2x^3+1}+\sqrt{2x^3-1}}$$

$$\lim_{x\to+\infty} \frac{\frac{2}{\sqrt{2x^3+1}+\sqrt{2x^3-1}}}{\frac{1}{x^{\frac{3}{2}}\sqrt{2}}} = \frac{2x^{\frac{3}{2}}\sqrt{2}}{x^{\frac{3}{2}}\left(\sqrt{2+\frac{1}{x^3}}+\sqrt{2-\frac{1}{x^3}}\right)} = 1$$

$$\int_1^{+\infty} \frac{1}{x^{\frac{3}{2}}\sqrt{2}}\ \mathrm{d}x \overset{\frac{3}{2}>1}{\to} \text{converges} \overset{\text{LCT}}{\Longrightarrow} \int_1^{+\infty} \sqrt{2x^3+1}-\sqrt{2x^3-1}\ \mathrm{d}x \to \text{converges}$$

i) $\int_1^{+\infty} \sqrt{2x^3+x+1}-\sqrt{2x^3-1}\ \mathrm{d}x$,

$$\sqrt{2x^3+x+1}-\sqrt{2x^3-1} = \frac{\sqrt{2x^3+x+1}-\sqrt{2x^3-1}}{\sqrt{2x^3+x+1}+\sqrt{2x^3-1}}\cdot(\sqrt{2x^3+x+1}+\sqrt{2x^3-1})$$

$$= \frac{2x^3+x+1-2x^3+1}{\sqrt{2x^3+1}+\sqrt{2x^3-1})} = \frac{2+x}{\sqrt{2x^3+x+1}+\sqrt{2x^3-1}}$$

$$\lim_{x\to+\infty} \frac{\frac{2+x}{\sqrt{2x^3+x+1}+\sqrt{2x^3-1}}}{\frac{1}{2x^{\frac{1}{2}}\sqrt{2}}} = \frac{2x^{\frac{3}{2}}\sqrt{2}\left(\frac{2}{x}+1\right)}{x^{\frac{3}{2}}\left(\sqrt{2+\frac{1}{x^2}+\frac{1}{x^3}}+\sqrt{2-\frac{1}{x^3}}\right)} = 1$$

$$\int_1^{+\infty} \frac{1}{2x^{\frac{1}{2}}\sqrt{2}}\ \mathrm{d}x \overset{\frac{1}{2}\leq 1}{\to} \text{diverges} \overset{\text{LCT}}{\Longrightarrow} \int_1^{+\infty} \sqrt{2x^3+x+1}-\sqrt{2x^3-1}\ \mathrm{d}x \to \text{diverges}$$

Exercise 2 (20 points).

Prove that the improper integral

$$\int_1^{+\infty} \frac{\sin(3x)}{\sqrt[4]{x}}\ \mathrm{d}x$$

is conditionally convergent

$$\int_1^{+\infty} \frac{\sin(3x)}{\sqrt[4]{x}}\ \mathrm{d}x \overset{u=3x}{=} \frac{1}{\sqrt[4]{3^3}}\int_3^{+\infty} \frac{\sin(u)}{u^{\frac{1}{4}}}\ \mathrm{d}u \overset{\text{IbP}}{=} \frac{1}{\sqrt[4]{3^3}}\int_3^{+\infty} \frac{1}{u^{\frac{1}{4}}}\ \mathrm{d}(-\cos u)$$

$$\overset{\text{IbP}}{=} \frac{1}{\sqrt[4]{3^3}}\left(\left[\frac{-\cos u}{u^{\frac{1}{4}}}\right]_3^{+\infty} - \frac{1}{4}\int_3^{+\infty}\frac{\cos u}{u^{\frac{5}{4}}}\ \mathrm{d}u\right)$$

$$\left|\frac{\cos u}{u^{\frac{1}{4}}}\right| \leq \frac{1}{u^{\frac{1}{4}}} \implies -\frac{1}{u^{\frac{1}{4}}} \leq \frac{\cos u}{u^{\frac{1}{4}}} \leq \frac{1}{u^{\frac{1}{4}}} \overset{u\to+\infty}{\Longrightarrow} 0 \leq \frac{\cos u}{u^{\frac{1}{4}}} \leq 0 \implies \lim_{u\to+\infty}\frac{\cos u}{u^{\frac{1}{4}}} = 0$$

$$\left[\frac{-\cos u}{u^{\frac{1}{4}}}\right]_3^{+\infty} = \lim_{u\to+\infty}\frac{\cos u}{u^{\frac{1}{4}}} - \frac{\cos 3}{3^{\frac{1}{4}}} = -\frac{\cos 3}{3^{\frac{1}{4}}} \in \mathbb{R}$$

$$\left|\frac{\cos u}{u^{\frac{5}{4}}}\right| \leq \frac{1}{u^{\frac{5}{4}}} \text{ and } \int_3^{+\infty}\frac{1}{u^{\frac{5}{4}}}\ \mathrm{d}u \overset{\frac{5}{4}>1}{\longrightarrow} \text{converges} \overset{\text{DCT}}{\Longrightarrow} \int_3^{+\infty}\left|\frac{\cos u}{u^{\frac{5}{4}}}\right|\ \mathrm{d}u \longrightarrow \text{converges}$$

$$\implies \int_3^{+\infty}\frac{\cos u}{u^{\frac{5}{4}}}\ \mathrm{d}u \longrightarrow \text{converges} \implies \int_1^{+\infty}\frac{\sin(3x)}{\sqrt[4]{x}}\ \mathrm{d}x \longrightarrow \text{converges}$$

$$\int_1^{+\infty}\left|\frac{\sin(3x)}{\sqrt[4]{x}}\right|\ \mathrm{d}x \overset{u=3x}{=} \frac{1}{\sqrt[4]{3^3}}\int_3^{+\infty}\left|\frac{\sin(u)}{u^{\frac{1}{4}}}\right|\ \mathrm{d}u \overset{|\sin x|\geq\sin^2 x}{\geq} \frac{1}{\sqrt[4]{3^3}}\int_3^{+\infty}\frac{\sin^2(u)}{u^{\frac{1}{4}}}\ \mathrm{d}u$$

$$= \frac{1}{2\sqrt[4]{3^3}}\int_3^{+\infty}\frac{1}{u^{\frac{1}{4}}}\ \mathrm{d}u - \frac{1}{2\sqrt[4]{3^3}}\int_3^{+\infty}\frac{\cos 2u}{u^{\frac{1}{4}}}\ \mathrm{d}u \overset{v=2u}{=} \frac{1}{2\sqrt[4]{3^3}}\int_3^{+\infty}\frac{1}{u^{\frac{1}{4}}}\ \mathrm{d}u - \frac{1}{\sqrt[4]{2^9\cdot 3^3}}\int_6^{+\infty}\frac{\cos v}{v^{\frac{1}{4}}}\ \mathrm{d}v$$

$$\int_6^{+\infty}\frac{\cos v}{v^{\frac{1}{4}}}\ \mathrm{d}v \overset{\text{IbP}}{=} \left[\frac{\sin v}{v^{\frac{1}{4}}}\right]_6^{+\infty} + \frac{1}{4}\int_6^{+\infty}\frac{\sin v}{v^{\frac{5}{4}}}\ \mathrm{d}v$$

$$\left|\frac{\sin v}{v^{\frac{1}{4}}}\right| \leq \frac{1}{v^{\frac{1}{4}}} \implies -\frac{1}{v^{\frac{1}{4}}} \leq \frac{\sin v}{v^{\frac{1}{4}}} \leq \frac{1}{v^{\frac{1}{4}}} \overset{v\to+\infty}{\Longrightarrow} 0 \leq \frac{\sin v}{v^{\frac{1}{4}}} \leq 0 \implies \lim_{v\to+\infty}\frac{\sin v}{v^{\frac{1}{4}}} = 0$$

$$\left[\frac{\sin v}{v^{\frac{1}{4}}}\right]_6^{+\infty} = \lim_{v\to+\infty} \frac{\sin v}{v^{\frac{1}{4}}} - \frac{\sin 6}{6^{\frac{1}{4}}} = -\frac{\sin 6}{6^{\frac{1}{4}}} \in \mathbb{R}$$

$$\left|\frac{\sin v}{v^{\frac{5}{4}}}\right| \leq \frac{1}{v^{\frac{5}{4}}} \text{ and } \int_6^{+\infty} \frac{1}{v^{\frac{5}{4}}}\,\mathrm{d}v \xrightarrow{\frac{5}{4}>1} \text{converges} \overset{\text{DCT}}{\Longrightarrow} \int_6^{+\infty} \left|\frac{\sin v}{v^{\frac{5}{4}}}\right| \mathrm{d}v \longrightarrow \text{converges}$$

$$\Longrightarrow \int_6^{+\infty} \frac{\sin v}{v^{\frac{5}{4}}}\,\mathrm{d}v \longrightarrow \text{converges} \Longrightarrow \int_6^{+\infty} \frac{\cos v}{v^{\frac{1}{4}}}\,\mathrm{d}v \longrightarrow \text{converges}$$

$$\int_3^{+\infty} \frac{1}{u^{\frac{1}{4}}}\,\mathrm{d}u \xrightarrow{\frac{1}{4}\leq 1} \text{diverges} \overset{\text{DCT}}{\Longrightarrow} \int_1^{+\infty} \left|\frac{\sin(3x)}{\sqrt[4]{x}}\right| \mathrm{d}x \longrightarrow \text{diverges} \Longrightarrow \text{conditionally convergent}$$

Exercise 3 (20 points).

Prove the convergence of the following improper integrals:

a) $\int_1^{+\infty} \frac{\cos(3x-1)}{\sqrt[3]{x^4}}\,\mathrm{d}x$;

$$\left|\frac{\cos(3x-1)}{\sqrt[3]{x^4}}\right| \leq \frac{1}{\sqrt[3]{x^4}} \quad \int_1^{+\infty} \frac{1}{x^{\frac{4}{3}}}\,\mathrm{d}x \xrightarrow{\frac{4}{3}>1} \text{converges} \overset{\text{DTC}}{\Longrightarrow} \int_1^{+\infty} \left|\frac{\cos(3x-1)}{\sqrt[3]{x^4}}\right| \mathrm{d}x \longrightarrow \text{converges}$$

$$\therefore \int_1^{+\infty} \frac{\cos(3x-1)}{\sqrt[3]{x^4}}\,\mathrm{d}x \longrightarrow \text{converges}$$

b) $\int_{10}^{+\infty} \frac{\ln\left((\ln x)^2+1\right)}{x\sqrt[4]{x}}\,\mathrm{d}x$.

$$\lim_{x\to+\infty} \frac{\ln\left((\ln x)^2+1\right)}{x^{\frac{1}{5}}} \overset{\text{L'Hôpital}}{=} \lim_{x\to+\infty} \frac{\frac{2\ln x}{x(\ln x)^2+1}\cdot}{\frac{1}{5x^{\frac{4}{5}}}} = \lim_{x\to+\infty} \frac{10\ln x}{x^{\frac{1}{5}}(\ln x)^2+1} \overset{\text{obviously}}{=} 0$$

$$\therefore \ln\left((\ln x)^2+1\right) \leq x^{\frac{1}{5}}$$

$$\frac{\ln\left((\ln x)^2+1\right)}{x\sqrt[4]{x}} = \frac{\ln\left((\ln x)^2+1\right)}{x^{\frac{5}{4}}} \leq \frac{x^{\frac{1}{5}}}{x^{\frac{5}{4}}} = \frac{1}{x^{\frac{21}{20}}}, \quad \int_{10}^{+\infty} \frac{1}{x^{\frac{21}{20}}}\,\mathrm{d}x \xrightarrow{\frac{21}{20}>1} \text{converges}$$

$$\therefore \text{by Direct Comparison Test} \int_{10}^{+\infty} \frac{\ln\left((\ln x)^2+1\right)}{x\sqrt[4]{x}}\,\mathrm{d}x \to \text{converges}$$