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CS101 Calculus 2 Section C - Homework 9

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Exercise 1 (40 points).

Determine the convergence of the following improper integrals of non-negative integrands:

a) $\int_0^1 \frac{e^{3\sqrt[3]{x}}-1}{x^2}\,\mathrm{d}x$;

$$\lim_{x\to 0}\frac{\frac{e^{3\sqrt[3]{x}}-1}{x^2}}{\frac{3}{x^{\frac{5}{3}}}} = \lim_{x\to 0}\frac{e^{3x^{\frac{1}{3}}}-1}{3x^{\frac{1}{3}}} \overset{u=3x^{\frac{1}{3}}}{=} \lim_{u\to 0}\frac{e^u-1}{u} = 1$$

$$\int_0^1 \frac{3}{x^{\frac{5}{3}}}\,\mathrm{d}x \xrightarrow{\frac{5}{3}\geq 1} \text{diverges} \overset{\text{LCT}}{\Longrightarrow} \int_0^1 \frac{e^{3\sqrt[3]{x}}-1}{x^2}\,\mathrm{d}x \longrightarrow \text{diverges}$$

b) $\int_0^3 \frac{\sqrt[5]{1+4x}-1}{\sqrt{x}\sin x}\,\mathrm{d}x$;

$$\lim_{x\to 0}\frac{\frac{\sqrt[5]{1+4x}-1}{\sqrt{x}\sin x}}{\frac{4}{5\sqrt{x}}} = \lim_{x\to 0}\frac{5}{4}\frac{\sqrt[5]{1+4x}-1}{\sin x} \overset{\text{L'Hôpital}}{=} \lim_{x\to 0}\frac{1}{\cos x(1+4x)^{\frac{4}{5}}} = 1$$

$$\int_0^3 \frac{4}{5\sqrt{x}}\,\mathrm{d}x \xrightarrow{\frac{1}{2}<1} \text{converges} \overset{\text{LCT}}{\Longrightarrow} \int_0^3 \frac{\sqrt[5]{1+4x}-1}{\sqrt{x}\sin x}\,\mathrm{d}x \longrightarrow \text{converges}$$

c) $\int_0^1 \frac{\ln\left(1+\sqrt{2x}\right)}{x(e^{\sqrt[3]{x}}-1)}\,\mathrm{d}x$;

$$\frac{\ln\left(1+\sqrt{2x}\right)}{x(e^{\sqrt[3]{x}}-1)} \leq \frac{\sqrt{2x}}{x(e^{\sqrt[3]{x}}-1)} = \frac{\sqrt{2}}{\sqrt{x}(e^{\sqrt[3]{x}}-1)}$$

$$\lim_{x\to 0}\frac{e^{\sqrt[3]{x}}-1}{x^{\frac{1}{3}}} = 1 \implies e^{\sqrt[3]{x}}-1 \sim x^{\frac{1}{3}},\ \text{at } x=0$$

$$\therefore \frac{\sqrt{2}}{\sqrt{x}(e^{\sqrt[3]{x}}-1)} \sim \frac{\sqrt{2}}{x^{\frac{5}{6}}} \implies \int_0^1 \frac{\sqrt{2}}{x^{\frac{5}{6}}}\,\mathrm{d}x \xrightarrow{\frac{5}{6}<1} \text{converges} \overset{\text{LCT}}{\Longrightarrow} \int_0^1 \frac{\sqrt{2x}}{x(e^{\sqrt[3]{x}}-1)}\,\mathrm{d}x \longrightarrow \text{converges}$$

$$\therefore \text{by Direct Comparison Test} \int_0^1 \frac{\ln\left(1+\sqrt{2x}\right)}{x(e^{\sqrt[3]{x}}-1)}\,\mathrm{d}x \longrightarrow \text{converges}$$

d) $\int_0^4 \frac{\sqrt[3]{\sin^2 x}}{\sqrt{x(4-x)}}\,\mathrm{d}x$.

$$\left|\frac{\sqrt[3]{\sin^2 x}}{\sqrt{x(4-x)}}\right| \le \frac{1}{\sqrt{x(4-x)}}$$

$$\int_0^4 \frac{1}{\sqrt{x(4-x)}}\,\mathrm{d}x \overset{x=2u^2}{=} \int_0^1 \frac{8u}{\sqrt{4-4u^2}\sqrt{4u^2}}\,\mathrm{d}u = 2\int_0^1 \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u \longrightarrow \text{can be solved}$$

$$\int_0^4 \frac{1}{\sqrt{x(4-x)}}\,\mathrm{d}x \longrightarrow \text{converges} \overset{\text{DCT}}{\Longrightarrow} \int_0^4 \left|\frac{\sqrt[3]{\sin^2 x}}{\sqrt{x(4-x)}}\right|\,\mathrm{d}x \longrightarrow \text{converges}$$

$$\therefore \int_0^4 \frac{\sqrt[3]{\sin^2 x}}{\sqrt{x(4-x)}}\,\mathrm{d}x \longrightarrow \text{converges}$$

Exercise 2 (30 points).

Determine the convergence of the following improper integrals:

a) $\int_0^{+\infty} \frac{\sin\frac{1}{\sqrt{x}}}{\sqrt[7]{x^6}}\,\mathrm{d}x$;

$$\int_0^{+\infty} \frac{\sin\frac{1}{\sqrt{x}}}{\sqrt[7]{x^6}}\,\mathrm{d}x = \int_0^1 \frac{\sin\frac{1}{\sqrt{x}}}{\sqrt[7]{x^6}}\,\mathrm{d}x + \int_1^{+\infty} \frac{\sin\frac{1}{\sqrt{x}}}{\sqrt[7]{x^6}}\,\mathrm{d}x$$

$$\lim_{x\to+\infty} \frac{\frac{\sin\frac{1}{\sqrt{x}}}{\sqrt[7]{x^6}}}{\frac{1}{x^{\frac{19}{14}}}} = \lim_{x\to+\infty} \frac{\sin\frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x}}} \overset{u=\frac{1}{\sqrt{x}}}{=} \lim_{u\to 0} \frac{\sin u}{u} = 1$$

$$\int_1^{+\infty} \frac{1}{x^{\frac{19}{14}}} \overset{\frac{19}{14}>1}{=\!=\!\longrightarrow} \text{converges} \Longrightarrow \int_1^{+\infty} \frac{\sin\frac{1}{\sqrt{x}}}{\sqrt[7]{x^6}}\,\mathrm{d}x \longrightarrow \text{converges}$$

$$\left|\frac{\sin\frac{1}{\sqrt{x}}}{\sqrt[7]{x^6}}\right| \le \frac{1}{\sqrt[7]{x^6}}\ \int_0^1 \frac{1}{x^{\frac{6}{7}}}\,\mathrm{d}x \overset{\frac{6}{7}<1}{\to} \text{converges} \Longrightarrow \int_0^1 \left|\frac{\sin\frac{1}{\sqrt{x}}}{\sqrt[7]{x^6}}\right|\,\mathrm{d}x \to \text{converges}$$

$$\therefore \int_0^1 \frac{\sin\frac{1}{\sqrt{x}}}{\sqrt[7]{x^6}}\,\mathrm{d}x \to \text{converges} \Longrightarrow \int_0^{+\infty} \frac{\sin\frac{1}{\sqrt{x}}}{\sqrt[7]{x^6}}\,\mathrm{d}x \to \text{converges}$$

b) $\int_3^{+\infty} \frac{\sqrt{2x+5}-\sqrt{2x-1}}{\sqrt[6]{(x-3)^5}}\,\mathrm{d}x$;

$$\int_3^{+\infty} \frac{\sqrt{2x+5}-\sqrt{2x-1}}{\sqrt[6]{(x-3)^5}}\,\mathrm{d}x = \int_3^{+\infty} \frac{6}{\sqrt[6]{(x-3)^5}(\sqrt{2x+5}+\sqrt{2x-1})}\,\mathrm{d}x$$

$$= \int_3^{10} \frac{6}{\sqrt[6]{(x-3)^5}(\sqrt{2x+5}+\sqrt{2x-1})}\,\mathrm{d}x + \int_{10}^{+\infty} \frac{6}{\sqrt[6]{(x-3)^5}(\sqrt{2x+5}+\sqrt{2x-1})}\,\mathrm{d}x$$

$$\lim_{x\to+\infty} \frac{\frac{6}{\sqrt[6]{(x-3)^5}(\sqrt{2x+5}+\sqrt{2x-1})}}{\frac{6}{\sqrt{2}x^{\frac{4}{3}}}} = 1,\ \int_{10}^{+\infty} \frac{6}{\sqrt{2}x^{\frac{4}{3}}}\,\mathrm{d}x \overset{\frac{4}{3}>1}{\longrightarrow} \text{converges}$$

$$\overset{\text{LCT}}{\therefore} \int_{10}^{+\infty} \frac{6}{\sqrt[6]{(x-3)^5}(\sqrt{2x+5}+\sqrt{2x-1})}\,\mathrm{d}x \longrightarrow \text{converges}$$

$$\frac{6}{\sqrt[6]{(x-3)^5}(\sqrt{2x+5}+\sqrt{2x-1})} \leq \frac{6}{\sqrt[6]{(x-3)^5}}$$

$$\int_3^{10} \frac{6}{\sqrt[6]{(x-3)^5}}\,\mathrm{d}x \overset{u=x-3}{=} \int_0^7 \frac{6}{u^{\frac{5}{6}}}\,\mathrm{d}u \xrightarrow{\frac{5}{6}<1} \text{converges} \implies$$

$$\overset{\text{DCT}}{\Longrightarrow} \int_3^{10} \frac{6}{\sqrt[6]{(x-3)^5}(\sqrt{2x+5}+\sqrt{2x-1})}\,\mathrm{d}x \longrightarrow \text{converges}$$

$$\therefore \int_3^{+\infty} \frac{\sqrt{2x+5}-\sqrt{2x-1}}{\sqrt[6]{(x-3)^5}}\,\mathrm{d}x \longrightarrow \text{converges}$$

c) $\int_1^{+\infty} \frac{\arctan(e^x)\cdot\cos\sqrt{x-1}}{x\sqrt{x+1}\sqrt[3]{\ln(x)}}\,\mathrm{d}x$.

$$\int_1^{+\infty} \frac{\arctan(e^x)\cdot\cos\sqrt{x-1}}{x\sqrt{x+1}\sqrt[3]{\ln(x)}}\,\mathrm{d}x = \int_1^{2} \frac{\arctan(e^x)\cdot\cos\sqrt{x-1}}{x\sqrt{x+1}\sqrt[3]{\ln(x)}}\,\mathrm{d}x + \int_2^{+\infty} \frac{\arctan(e^x)\cdot\cos\sqrt{x-1}}{x\sqrt{x+1}\sqrt[3]{\ln(x)}}\,\mathrm{d}x$$

$$\frac{\arctan(e^x)\cdot\cos\sqrt{x-1}}{x\sqrt{x+1}\sqrt[3]{\ln(x)}} \overset{x=1}{=} \frac{\arctan(e)}{\sqrt{2}\sqrt[3]{\ln(x)}} = \frac{\arctan e}{\sqrt{2}} \cdot \frac{1}{\ln^{\frac{1}{3}}(x)}$$

$$\int_1^{2} \frac{1}{\ln^{\frac{1}{3}}(x)}\,\mathrm{d}x \longrightarrow \text{converges} \implies \int_1^{2} \frac{\arctan(e^x)\cdot\cos\sqrt{x-1}}{x\sqrt{x+1}\sqrt[3]{\ln(x)}}\,\mathrm{d}x \longrightarrow \text{converges}$$

$$\left|\frac{\arctan(e^x)\cdot\cos\sqrt{x-1}}{x\sqrt{x+1}\sqrt[3]{\ln(x)}}\right| \leq \frac{\arctan(e^x)}{x\sqrt{x+1}}$$

$$\lim_{x\to+\infty} \frac{\frac{\arctan(e^x)}{x\sqrt{x+1}}}{\frac{\frac{\pi}{2}}{x^{\frac{3}{2}}}} \overset{\text{magically}}{=} 1 \int_2^{+\infty} \frac{\frac{\pi}{2}}{x^{\frac{3}{2}}}\,\mathrm{d}x \xrightarrow{\frac{3}{2}>1} \text{converges} \overset{\text{LCT}}{\Longrightarrow} \int_2^{+\infty} \frac{\arctan(e^x)}{x\sqrt{x+1}}\,\mathrm{d}x \longrightarrow \text{converges}$$

$$\overset{\text{DCT}}{\therefore} \int_2^{+\infty} \left|\frac{\arctan(e^x)\cdot\cos\sqrt{x-1}}{x\sqrt{x+1}\sqrt[3]{\ln(x)}}\right|\,\mathrm{d}x \longrightarrow \text{converges}$$

$$\therefore \int_2^{+\infty} \frac{\arctan(e^x)\cdot\cos\sqrt{x-1}}{x\sqrt{x+1}\sqrt[3]{\ln(x)}}\,\mathrm{d}x \longrightarrow \text{converges}$$

$$\therefore \int_1^{+\infty} \frac{\arctan(e^x)\cdot\cos\sqrt{x-1}}{x\sqrt{x+1}\sqrt[3]{\ln(x)}}\,\mathrm{d}x \longrightarrow \text{converges}$$

Exercise 3 (30 points).

Find the sum of the series or prove that it diverges using the partial sums:

a) $\sum_{n=1}^{\infty} \frac{(-1)^n+2^{2n}}{8^n}$;

$$\sum_{n=1}^{\infty} \frac{(-1)^n+2^{2n}}{8^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{8^n} + \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$s_1 = a_1 = -\frac{1}{8} \approx -0.125, s_2 = s_1 + a_2 = -\frac{1}{8} + \frac{1}{8^2} = -\frac{7}{64} \approx -0.109$$

$$s_3 = s_2 + a_3 = -\frac{7}{64} - \frac{1}{8^3} = -\frac{57}{512} \approx -0.1113, s_4 = s_3 + a_4 = -\frac{57}{512} + \frac{1}{8^4} = -\frac{455}{4096} \approx -0.1111$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n + 2^{2n}}{8^n} = -\frac{1}{9}$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \overset{\text{obviously}}{=} 1 \implies \sum_{n=1}^{\infty} \frac{(-1)^n + 2^{2n}}{8^n} = \frac{8}{9}$$

b) $\sum_{n=1}^{\infty} \frac{1}{(3n+1)(3n+4)}$;

$$\frac{1}{(3n+1)(3n+4)} \overset{\text{partial fractions}}{=} \frac{1}{3(3n+1)} - \frac{1}{3(3(n+1)+1)}$$

$$\sum_{n=1}^{\infty} \frac{1}{(3n+1)(3n+4)} = \frac{1}{12} - \frac{1}{21} + \frac{1}{21} - \frac{1}{30} + ... + \frac{1}{3(3n+1)} - \frac{1}{3(3(n+1)+1)}$$

$$= \frac{1}{12} - \cancel{\frac{1}{21}} + \cancel{\frac{1}{21}} - \cancel{\frac{1}{30}} + ... + \cancel{\frac{1}{3(3n+1)}} - \frac{1}{3(3(n+1)+1)} = \frac{1}{12} - \lim_{n\to+\infty} \frac{1}{3(3(n+1)+1)} = \frac{1}{12}$$

c) $\sum_{n=1}^{\infty} \ln\left(1 + \frac{3}{3n-2}\right)$;

$$a_n = \ln\left(1 + \frac{3}{3n-2}\right) = \ln\left(\frac{3n+1}{3n-2}\right) = \ln(3n+1) - \ln(3(n-1)+1)$$

$$\sum_{n=1}^{\infty} \ln\left(1 + \frac{3}{3n-2}\right) = \ln 4 - 0 + \ln 7 - \ln 4 + ... + \ln(3n+1) - \ln(3(n-1)+1)$$

$$= \cancel{\ln 4} - 0 + \cancel{\ln 7} - \cancel{\ln 4} + ... + \ln(3n+1) - \cancel{\ln(3(n-1)+1)} = \lim_{n\to+\infty} \ln(3n+1) \longrightarrow \text{diverges}$$

d) $\sum_{n=1}^{\infty} \left(\left(3 - \frac{2}{n}\right)^n + \frac{2}{7^n}\right)$.

$$s_1 = a_1 = \left(\left(3 - \frac{2}{1}\right)^1 + \frac{2}{7^1}\right) = \frac{9}{7}$$

$$s_2 = s_1 + a_2 = \frac{9}{7} + \left(\left(3 - \frac{2}{2}\right)^2 + \frac{2}{7^2}\right) = \frac{198}{49} = \frac{261}{49}$$

Partial sums become larger and larger $\implies$ diverges