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CS102 Calculus 3 Section G - Homework 10

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Problem 1 (20 points).

Find a parametric representation for the surface.

a) The plane through the point $(1, 2, 3)$ that contains the vectors $\hat{\mathbf{i}} - \hat{\mathbf{j}}$ and $\hat{\mathbf{j}} - \hat{\mathbf{k}}$.

$$\mathbf{r} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + s \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, s, t \in \mathbb{R}$$

b) The part of the hyperboloid $4x^2 - 4y^2 - z^2 = 4$ that lies in the semispace given by $x > 0$.

$$\mathbf{r} = \begin{bmatrix} \sqrt{1 + s^2 + \frac{t^2}{4}} \\ s \\ t \end{bmatrix}, s, t \in \mathbb{R}$$

c) The part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$.

$$x^2 + y^2 + x^2 + y^2 \leq 4 \implies x^2 + y^2 \leq 2$$

$$\mathbf{r} = \begin{bmatrix} \sqrt{2}s\cos(t) \\ \sqrt{2}s\sin(t) \\ \sqrt{4 - 2s^2} \end{bmatrix}, s \in [0, 1], t \in [0, 2\pi]$$

d) The part of the cylinder $y^2 + z^2 = 16$ that lies between the planes $x = 0$ and $x = 5$.

$$\mathbf{r} = \begin{bmatrix} s \\ 4\cos(t) \\ 4\sin(t) \end{bmatrix}, s \in [0, 5], t \in [0, 2\pi]$$

Problem 2 (20 points).

Find an equation of the tangent plane to the given parametric surface at the specified point P.

a) $x = u + v$, $y = 3u^2$, $z = u - v$, where $P = (2, 3, 0)$.

$$u = v = 1, \mathbf{r}'_u = \{1, 6u, 1\} = \{1, 6, 1\}, \mathbf{r}'_v = \{1, 0, -1\}, \mathbf{r}'_u \times \mathbf{r}'_v = \{-6, 2, -6\}$$

$$\mathbf{r}_{\text{tangent}} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 6 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, s, t \in \mathbb{R} \text{ or } 6x - 2y + 6z = 6$$

b) $\mathbf{r}(u, v) = \{u\cos(v), u\sin(v), v\}$, where P is the point corresponding to $u = 1$, $v = \pi$.

$$\mathbf{r}'_u = \{\cos(v), \sin(v), 0\} = \{-1, 0, 0\}, \mathbf{r}'_v = \{-u\sin(v), u\cos(v), 1\} = \{0, -1, 1\}6, \mathbf{r}'_u \times \mathbf{r}'_v = \{0, 1, 1\}$$

$$\mathbf{r}_{\text{tangent}} = \begin{bmatrix} -1 \\ 0 \\ \pi \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, s, t \in \mathbb{R} \text{ or } y + z = \pi$$

Problem 3 (30 points).

Find the area of the surface.

a) The part of the plane $3x + 2y + z = 6$ that lies in the first octant.

$$z = 6 - 3x - 2y, x \in [0, 2], y \in [0, 3]$$

$$S = \int_0^2 \int_0^{3-\frac{3}{2}x} \sqrt{1 + 9 + 4}\,\mathrm{d}y\,\mathrm{d}x = \sqrt{14} \int_0^2 3 - \frac{3}{2}x\,\mathrm{d}x = 3\sqrt{14}$$

b) The part of the plane $x + 2y + 3z = 1$ that lies inside the cylinder $x^2 + y^2 = 3$.

$$z = \frac{1}{3} - \frac{1}{3}x - \frac{2}{3}y$$

$$S = \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{1 + \frac{1}{9} + \frac{4}{9}} r\,\mathrm{d}r\,\mathrm{d}\varphi = \pi\sqrt{14}$$

c) The surface $z = \dfrac{2}{3}(x^{3/2} + y^{3/2})$, $0 \le x \le 1$, $0 \le y \le 1$.

$$S = \int_0^1 \int_0^1 \sqrt{1 + x + y}\,\mathrm{d}y\,\mathrm{d}x = \frac{2}{3} \int_0^1 (2 + x)^{3/2} - (1 + x)^{3/2}\,\mathrm{d}x = \frac{4}{15}(9\sqrt{3} - 8\sqrt{2} + 1)$$

d) The part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

$$S = \int_0^{2\pi} \int_0^1 \sqrt{1 + rr}\,\mathrm{d}r\,\mathrm{d}\varphi = \int_0^{2\pi} \int_1^2 \sqrt{u}(u - 1)\,\mathrm{d}u\,\mathrm{d}\varphi = 8\pi\frac{\sqrt{2} + 1}{15}$$

e) The part of the surface $y = 4x + z^2$ that lies between the planes $x = 0$, $x = 1$, $z = 0$, and $z = 1$.

$$S = \int_0^1 \int_0^1 \sqrt{1 + 16 + 4z^2}\,\mathrm{d}z\,\mathrm{d}x \overset{*}{=} \frac{17\sqrt{17}}{4} \left(\operatorname{arcsinh}\left(\sqrt{\frac{17}{4}} \right) + \sqrt{\frac{357}{4}} \right)$$

$$* \int \sqrt{17 + 4z^2}\,\mathrm{d}z \overset{z=\sqrt{\frac{17}{4}}\sinh\varphi}{=} \frac{17\sqrt{17}}{2} \int \cosh^2\varphi\,\mathrm{d}\varphi = \frac{17\sqrt{17}}{4} \left(\operatorname{arcsinh}\left(\sqrt{\frac{17}{4}} z \right) + \sqrt{\frac{17}{4}} z \sqrt{1 + \frac{17}{4} z^2} \right)$$

f) The surface with parametric equations $x = u^2$, $y = uv$, $z = \frac{1}{2}v^2$, $0 \le u \le 1$, $0 \le v \le 2$.

$$S = \int_0^1 \int_0^2 \|\{2u, v, 0\} \times \{0, u, v\}\| \,\mathrm{d}v\,\mathrm{d}u = \int_0^1 \int_0^2 \sqrt{v^4 + 4u^2v^2 + 4u^4}\,\mathrm{d}v\,\mathrm{d}u$$

$$= \int_0^1 \int_0^2 v^2 + 2u^2 \,\mathrm{d}v\,\mathrm{d}u = \int_0^1 \frac{8}{3} + 4u^2 \,\mathrm{d}u = \frac{12}{3} = 4$$

Problem 4 (10 points).

Find the surface area of the solid, which is the intersection of cylinders $y^2 + z^2 = 1$ and $x^2 + z^2 = 1$.

We have four equal parts. Find are of one of the surfaces and multiply by 4. The parametrization is trivial:

$$\mathbf{r} = \{x, y, z\}, \text{ s.t. } x = \cos\theta, y = \cos\theta\tan\varphi, z = \sin\theta, \varphi \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right], \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\mathbf{r}'_\varphi = \begin{bmatrix} 0 \\ \cos\theta\sec^2\varphi \\ 0 \end{bmatrix}, \mathbf{r}'_\theta = \begin{bmatrix} -\sin\theta \\ -\sin\theta\tan\varphi \\ \cos\theta \end{bmatrix}, \mathbf{r}'_\varphi \times \mathbf{r}'_\theta = \begin{bmatrix} \cos^2\theta\sec^2\varphi \\ 0 \\ -\sin\theta\cos\theta\sec^2\varphi \end{bmatrix}$$

$$S_{0.25} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2\varphi\cos\theta \,\mathrm{d}\varphi\,\mathrm{d}\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos\theta\,\mathrm{d}\theta = 4 \implies S_{\text{total}} = 16$$

Problem 5 (20 points).

a) Find a parametric representation for the torus obtained by rotating about the z axis the circle in the $x - z$ plane with center $(b, 0, 0)$ and radius $a < b$.

$$\text{Parametric equation for circle: } \begin{cases} x = b + a\cos\varphi \\ z = a\sin\varphi \end{cases}, \varphi \in [0, 2\pi]$$

$$\therefore \mathbf{r}(\varphi, \theta) = \{x, y, z\}, \text{ s.t. } \begin{cases} x = (b + a\cos\varphi)\cos\theta \\ y = (b + a\cos\varphi)\sin\theta \\ z = a\sin\varphi \end{cases}, \varphi \in [0, 2\pi], \theta \in [0, 2\pi]$$

b) Use the parametric representation from part a) to find the surface area of the torus.

$$\mathbf{r}'_\varphi = \begin{bmatrix} -a\sin\varphi\cos\theta \\ -a\sin\varphi\sin\theta \\ a\cos\varphi \end{bmatrix}, \mathbf{r}'_\theta = \begin{bmatrix} -(b + a\cos\varphi)\sin\theta \\ (b + a\cos\varphi)\cos\theta \\ 0 \end{bmatrix}$$

$$\mathbf{r}'_\varphi \times \mathbf{r}'_\theta = \begin{bmatrix} -a\cos\varphi(b + a\cos\varphi)\cos\theta \\ -a\cos\varphi(b + a\cos\varphi)\sin\theta \\ -a\sin\varphi\cos\theta(b + a\cos\varphi)\cos\theta - a\sin\varphi\sin\theta(b + a\cos\theta)\sin\theta \end{bmatrix} = \begin{bmatrix} -a\cos\varphi(b + a\cos\varphi)\cos\theta \\ -a\cos\varphi(b + a\cos\varphi)\sin\theta \\ -a\sin\varphi(b + a\cos\varphi) \end{bmatrix}$$

$$\|\mathbf{r}'_\varphi \times \mathbf{r}'_\theta\| = \sqrt{a^2\cos^2\varphi(b + a\cos\varphi)^2 + a^2\sin^2\varphi(b + a\cos\varphi)^2} = a(b + a\cos\varphi)$$

$$S = \int_0^{2\pi} \int_0^{2\pi} ab + a^2\cos\varphi \,\mathrm{d}\varphi\,\mathrm{d}\theta = \int_0^{2\pi} 2\pi ab\,\mathrm{d}\theta = 4\pi^2 ab$$