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CS102 Calculus 3 Section G - Homework 11

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Problem 1 (20 points).

Evaluate the surface integral

a) $\iint_S (x+y+z)\,\mathrm{d}S$, where S is the parallelogram with parametric equations $x = u+v$, $y = u-v$, $z = 1+2u+v$, $0 \le u \le 2$, $0 \le v \le 1$.

$$\{1,1,2\} \times \{1,-1,1\} = \{3,1,-2\}$$

$$\int_0^2 \int_0^1 \sqrt{14}(4u+v+1)\,\mathrm{d}v\,\mathrm{d}u = \sqrt{14}\int_0^2 4u + \frac{3}{2}\,\mathrm{d}u = 11\sqrt{14}$$

b) $\iint_S y\,\mathrm{d}S$, where S is the helicoid with vector equation $\mathbf{r}(u,v) = \{2uv, u^2-v^2, u^2+v^2\}$, $0 \le u \le 1$, $0 \le v \le \pi$.

$$\{2v, 2u, 2u\} \times \{2u, -2v, 2v\} = \{8uv, 4u^2-4v^2, -4u^2-4v^2\}$$

$$\int_0^1 \int_0^\pi u^2 - v^2\sqrt{64u^2v^2 + 16(u^2-v^2)^2 + 16(u^2+v^2)^2}\,\mathrm{d}v\,\mathrm{d}u = \int_0^1 \int_0^\pi 4\sqrt{2}(u^2-v^2)(u^2+v^2)\,\mathrm{d}v\,\mathrm{d}u$$

$$= 4\sqrt{2}\int_0^1 \int_0^\pi u^4 - v^4\,\mathrm{d}v\,\mathrm{d}u = \frac{(\pi-\pi^5)4\sqrt{2}}{5}$$

c) $\iint_S y^2\,\mathrm{d}S$, where S is the part of the sphere $x^2+y^2+z^2=4$ that lies inside the cylinder $x^2+y^2=1$ and above the xy plane.

$$\because z \ge 0, z = \sqrt{4-x^2-y^2}, z'_x(x,y) = \frac{-x}{\sqrt{4-x^2-y^2}}, z'_y(x,y) = \frac{-y}{\sqrt{4-x^2-y^2}}$$

$$\int_0^{2\pi} \int_0^1 r^2 \sin^2\varphi \sqrt{\frac{r^2}{4-r^2}+1}r\,\mathrm{d}r\,\mathrm{d}\varphi = 2\pi\int_0^1 r^3\sqrt{\frac{1}{4-r^2}}\,\mathrm{d}r = \pi\int_3^4 (4-t)\sqrt{\frac{1}{t}}\,\mathrm{d}t$$

$$= \pi\int_3^4 4t^{-\frac{1}{2}} - t^{\frac{1}{2}}\,\mathrm{d}t = \pi\left[8t^{\frac{1}{2}} - \frac{2}{3}t^{\frac{3}{2}}\right] = \pi\left(16 - \frac{16}{3} - 8\sqrt{3} + 2\sqrt{3}\right) = \pi\left(\frac{32}{3} - 6\sqrt{3}\right)$$

d) $\iint_S (z + x^2y)\,\mathrm{d}S$, where S is the part of the cylinder $y^2+z^2=1$ that lies between the planes $x=0$ and $x=3$ in the first octant.

$$\mathbf{r} = (u, \cos(v), \sin(v)), u \in [0,3], v \in [0, 2\pi], \{1,0,0\} \times \{0, -\sin(v), \cos(v)\} = \{0, -\cos(v), -\sin(v)\}$$

$$\int_0^{2\pi} \int_0^3 (\sin(v) + u^2\cos(v))\sqrt{1}\,\mathrm{d}u\,\mathrm{d}v = \int_0^{2\pi} 3\sin(v) + 9\cos(v)\,\mathrm{d}v = 0$$

Problem 2 (20 points).

Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$ for the given vector field $\mathbf{F}$ and the oriented surface S (i.e. find the flux of $\mathbf{F}$ across S). For closed surfaces , use the positive (outward) orientation.

a) $F(x, y, z) = \{ze^{xy}, -3ze^{xy}, xy\}$, where S is the parallelogram from Problem 1 a) with upward orientation.

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_0^2 \int_0^1 3(ze^{xy}) + (-3ze^{zy}) - 2((u+v)(u-v))\,\mathrm{d}v\,\mathrm{d}u = \int_0^2 \int_0^1 u^2 - v^2\,\mathrm{d}v\,\mathrm{d}u = 2$$

b) $F(x, y, z) = \{xy, yz, zx\}$, where S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \le x \le 1$, $0 \le y \le 1$ and has upward orientation.

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_0^1 \int_0^1 2x^3y + 8y^3 - 4y^3x^2 - y^5 + 4x - x^3 - xy^2\,\mathrm{d}y\,\mathrm{d}x = \frac{10}{3}$$

c) $F(x, y, z) = \{x, -z, y\}$, where S is the part of the sphere $x^2 + y^2 + z^2 = 4$ in the first octant, with orientation towards the origin.

$$z = \sqrt{4 - x^2 - y^2}, x, y \ge 0$$

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_0^{2\pi} \int_0^2 \frac{r^3 \cos^2 \varphi}{\sqrt{4 - r^2}}\,\mathrm{d}r\,\mathrm{d}\varphi = \frac{\pi}{2} \int_0^4 \frac{4 - t}{\sqrt{t}}\,\mathrm{d}t = \frac{\pi}{2} \int_0^4 4t^{-\frac{1}{2}} - t^{\frac{1}{2}}\,\mathrm{d}t = \frac{16\pi}{3}$$

d) $F(x, y, z) = \{x, 2y, 3z\}$, where S is the surface of the cube with vertices $(\pm 1, \pm 1, \pm 1)$.

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_E (1 + 2 + 3)\,\mathrm{d}V = 48$$

Problem 3 (30 points).

Use Stokes' Theorem to evaluate $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$. In each case C is oriented counterclockwise as viewed from above.

a) $F(x, y, z) = \{x+y^2, y+z^2, z+x^2\}$, where C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

$$\nabla \times \mathbf{F} = \{-2z, -2x, -2y\}, z = 1 - x - y$$

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \int_0^1 \int_0^{1-x} -2 + 2x - 2y\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 2(x-1)(1-x) - (1-x)^2\,\mathrm{d}x = -1$$

b) $F(x, y, z) = \{x^2 z, xy^2, z^2\}$, where C is the curve of intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$.

$$\nabla \times \mathbf{F} = \{0, x^2, y^2\}, z = 1 - x - y, x = r\cos\varphi, y = r\sin\varphi, r \in [0, 3], \varphi \in [0, 2\pi]$$

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \int_0^{2\pi} \int_0^3 r^2 \cos^2 \varphi + r^2 \sin^2 \varphi r\,\mathrm{d}r\,\mathrm{d}\varphi = \frac{3^4 \pi}{2}$$

c) $F(x, y, z) = \{yz, 2xz, e^{xy}\}$, where C is the circle $x^2 + y^2 = 16$, $z = 5$.

$$\nabla \times \mathbf{F} = \{xe^{xy} - 2x, y - ye^{xy}, z\}, \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \iint_D 5\,\mathrm{d}S = 80\pi$$

Problem 4 (30 points).

Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S}$; that is find the flux of $\mathbf{F}$ across S.

a) $F(x, y, z) = \{xye^z, xy^2z^3, -ye^z\}$, where S is the surface of the box bounded by the coordinate planes and the planes $x = 3$, $y = 2$, and $z = 1$.

The components of $\mathbf{F}$ have continous partial derivatives

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_0^3 \int_0^2 \int_0^1 ye^z + 2xyz^3 - ye^z \,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x = 2 \cdot \frac{2^2}{2} \cdot \frac{3^2}{2} \cdot \frac{1}{4} = \frac{9}{2}$$

b) b) $F(x, y, z) = \{3xy^2, xe^z, z^3\}$, where S is the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$.

The components of $\mathbf{F}$ have continous partial derivatives

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_E 3y^2 + 0 + 3z^2 \,\mathrm{d}V = \int_{-1}^2 \int_0^{2\pi} \int_0^1 3r^2 r \,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}x = \frac{9\pi}{2}$$

c) $F(x, y, z) = \{(\cos z + xy^2), xe^{-z}, (\sin y + x^2 z)\}$, where S is the surface of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

The components of $\mathbf{F}$ have continous partial derivatives

$$\iint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \iiint_E y^2 + 0 + x^2 \,\mathrm{d}V = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r^2 r \,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\varphi = 2\pi \int_0^2 4r^3 - r^5 \,\mathrm{d}r = 2\pi \left(2^4 - \frac{2^6}{6}\right)$$