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CS102 Calculus 3 Section G - Homework 1

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Problem 1 (10 points).

Assume $A = (1, 1, 0)$, $B = (2, 1, 1)$ and $C = (0, -1, a)$. Find a, if the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$ is $\frac{\pi}{3}$

$$\overrightarrow{AB} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ and } \overrightarrow{AB} = \begin{bmatrix} -1 \\ -2 \\ a \end{bmatrix}$$

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = 1 \cdot (-1) + 0 \cdot (-2) + 1 \cdot a = \sqrt{2} \cdot \sqrt{5 + a^2} \cos \frac{\pi}{3}$$

$$a - 1 = \sqrt{2} \cdot \sqrt{5 + a^2} \cos \frac{\pi}{3} \implies a \geq 1$$

$$(a-1)^2 = \frac{1}{2}(5 + a^2) \implies 2a^2 - 4a + 2 = 5 + a^2$$

$$a^2 - 4a - 3 = 0 \implies \begin{cases} a = 2 - \sqrt{7} < 1 \\ a = 2 + \sqrt{7} \end{cases} \implies a = 2 + \sqrt{7}$$

Problem 2 (10 points).

Let $\mathbf{x} = (0, 2, 3)$ and $\mathbf{y} = (1, 0, 1)$. Find a, if it is know that the vectors $\mathbf{x} + 3\mathbf{y}$ and $\mathbf{x} - a\mathbf{y}$ are orthogonal.

$$\mathbf{x} + 3\mathbf{y} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix} \text{ and } \mathbf{x} - a\mathbf{y} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} - a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -a \\ 2 \\ 3 - a \end{bmatrix}$$

$$(\mathbf{x} + 3\mathbf{y}) \cdot (\mathbf{x} - a\mathbf{y}) = -3a + 4 + 18 - 6a = 0 \implies a = \frac{22}{9}$$

Problem 3 (20 points).

Are the vectors $\mathbf{u} = (1, 2, -4)$, $\mathbf{v} = (-5, 3, -7)$ and $\mathbf{w} = (-1, 4, 2)$ in the same plane ?

$$\text{Take } \mathbf{x} = \mathbf{u} \times \mathbf{v} = \begin{bmatrix} -2 \\ 27 \\ 13 \end{bmatrix} \implies \mathbf{x} \perp \mathbf{u} \text{ and } \mathbf{x} \perp \mathbf{v}$$

$$\text{If } \mathbf{u}, \mathbf{v}, \mathbf{w} \text{ are in the same plane } \implies \text{ they should all be perpendicular } \mathbf{x}$$

$$\mathbf{w} \cdot \mathbf{x} \neq 0 \implies \mathbf{w} \not\perp \mathbf{x} \implies \text{ not in the same plane}$$

Problem 4 (30 points).

Prove that

a) $S\left((2,-1,2);5\right) \subset B\left((0,0,0);10\right)$.

$$A := (2,-1,2) \implies \left\|\overrightarrow{OA}\right\| = \sqrt{4+1+4} = 3$$

$$\because 3+5 < 10 \implies S\left((2,-1,2);5\right) \subset B\left((0,0,0);10\right)$$

b) the set $A = \{(x,y)|2 \leq x \leq 4, 0 \leq y \leq x^2\}$ is bounded.

Take upper and lower bounds of variables: $2 \leq x \leq 4$ and $0 \leq y \leq 16$

$$\forall b \in A, \|b\| \leq \sqrt{16^2+4^2} < 20 \implies A \subset B((0,0);20) \implies A \text{ is bounded}$$

Problem 5 (30 points).

Find $\text{int}(X)$ and $\partial(X)$ for the given set X. Determine whether X is open or closed ?

a) $X = \mathbb{Z} \times \mathbb{Z} = \{(x,y)|x,y \in \mathbb{Z}\} \subset \mathbb{R}^2$

$$\because \nexists \varepsilon > 0, x,y \in \mathbb{Z}, B((x,y);\varepsilon) \in X \implies \text{int}(X) = \varnothing$$

$$\because \exists \varepsilon > 0, x,y \in \mathbb{Z}, (x,y),(x+\varepsilon,y) \in B((x,y);\varepsilon) \text{ s.t. } (x,y) \in X, (x+\varepsilon,y) \notin X \implies \partial(X) = X$$

$$\therefore X \text{ is a closed set}$$

b) $X = \{(x,y)||x| < 1\} \subset \mathbb{R}^2$

$$\text{int}(X) = \{(x,y)|x \in (-1,1)\}, x,y \in \mathbb{R} \implies \text{int}(X) = X$$

$$\partial(X) = \{(x,y)|x = \pm 1\}, x,y \in \mathbb{R} \implies \partial(X) \notin X$$

$$\therefore X \text{ is an open set}$$

c) $X = \left\{(x,y,z)|1 < x^2+y^2+z^2 \leq 4\right\} \subset \mathbb{R}^3$
We have a set of points, which fall inside a sphere with radius 2 centered at the origin, but outside a sphere with radius 1 centered at the origin, including the points of the first sphere and not including the points of the second sphere.

$$\text{int}(X) = \left\{(x,y,z)|1 < x^2+y^2+z^2 < 4\right\}, x,y,z \in \mathbb{R}$$

$$\partial(X) = \left\{(x,y,z)|x^2+y^2+z^2 = 4\right\} \cup \left\{(x,y,z)|x^2+y^2+z^2 = 1\right\}, x,y,z \in \mathbb{R}$$

$$\because \partial(X) \notin X \text{ and } \text{int}(X) \neq X \implies X \text{ is nether an open, nor a closed set}$$