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CS102 Calculus 3 Section G - Homework 2

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Problem 1 (30 points).

For the given set X determine whether it is compact or not? Whenever X is not compact, bring an example of a sequence of points in X that does not contain a convergent subsequence in X.

a) X is a union of 2024 closed balls in $\mathbb{R}^2$.

$$\text{Take } \overline{A_k}(O_k, R_k), \text{ such that } X = \bigcup_{k=1}^{2024} A_k, O_k \in \mathbb{R}^2, R_k \in \mathbb{R}$$

X is bounded, since it is a union of closed balls, which are bounded. X is also closed, as a union of closed balls also contains all the boundary points.

$$\therefore X \text{ is compact}$$

b) $X = \{(x, y, z)|x^4 + y^4 + z^4 \leq 1\} \subset \mathbb{R}^3$

$$X \text{ is bounded by } \{(x, y, z)|x^4 + y^4 + z^4 = 1\} \subset \mathbb{R}^3 \text{ and includes the boundary points}$$

$$\therefore X \text{ is bounded and closed} \implies X \text{ is compact}$$

c) $X = \overline{B}(O, 7) \backslash \overline{B}(P, 1) \subset \mathbb{R}^3$, where $O = (0, 0, 0)$ and $P = (1, 2, 1)$.

We have the set of all the points that are at distance 7 from the origin, but that are not at distance 1 from the point $(1, 2, 1)$. The boundary points of $\overline{B}(P, 1) \subset \mathbb{R}^3$ are also boundary points of X, however they are not included in the set X, meaning that the set is not closed, therefore the set is not compact. Take the sequence $\{x_n\}_{n=1}^{\infty}$, where $x_n \in X, x_n \to (1, 2, 2)$. Take the subsequence of points $\{x_n\}_{n=2}^{\infty}, x_n \to (1, 2, 2) \notin X$, therefore X is not compact.

Problem 2 (20 points).

Which of the following sequences are bounded? Prove your claims.

a) $x_k = \left(\dfrac{k+1}{3k}, \dfrac{1}{2+k}\right)$;

$$\frac{k+1}{3k} = \frac{1}{3} + \frac{1}{3k} \leq \frac{2}{3}\, \forall k \in \mathbb{N} \text{ and } \frac{1}{2+k} \leq \frac{1}{3}\, \forall k \in \mathbb{N} \implies x_k \leq \left(\frac{2}{3}, \frac{1}{3}\right) \implies x_k \text{ is bounded}$$

b) $x_k = \left(\frac{2}{1-\sqrt[k]{k}}, 0, \cos k\right)$

$$\lim_{k\to\infty} \sqrt[k]{k} = 1 \implies \lim_{k\to\infty} \frac{2}{1-\sqrt[k]{k}} \to -o\infty \implies x_k \text{ is unbounded}$$

Problem 3 (20 points).

Prove, using the proposition that $\mathbf{x}_k \to \mathbf{a}$ iff $\|\mathbf{x_k} - a\| \to 0$, that

$$\lim_{k\to\infty} \left(\frac{3k^2-k+2}{k^2+5k+1}, \frac{k}{k+3}\right) = (3,1)$$

$$\lim_{k\to\infty} \|\mathbf{x_k} - a\| = \lim_{k\to\infty} \left\| \frac{3k^2-k+2}{k^2+5k+1} - 3, \frac{k}{k+3} - 1 \right\| = \lim_{k\to\infty} \left\| \frac{-16k-1}{k^2+5k+1}, \frac{-3}{k+3} \right\|$$

$$= \lim_{k\to\infty} \left\| \frac{-\frac{16}{k} - \frac{1}{k^2}}{1+\frac{5}{k}+\frac{1}{k^2}}, \frac{-\frac{3}{k}}{1+\frac{3}{k}} \right\| \overset{\lim_{x\to\infty} \frac{1}{x^n}=0, n>0}{=} 0 \implies \lim_{k\to\infty} \left(\frac{3k^2-k+2}{k^2+5k+1}, \frac{k}{k+3}\right) = (3,1)$$

Problem 4 (30 points).

Calculate the limit:

a) $\lim_{k\to\infty} \left(\frac{k+\sqrt{k}}{2k-\sqrt[3]{k}+1}, \left(\frac{2}{3}\right)^k, k\sin\left(\frac{1}{k}\right)\right)$

$$\lim_{k\to\infty} \frac{k+\sqrt{k}}{2k-\sqrt[3]{k}+1} = \lim_{k\to\infty} \frac{1+\frac{1}{k^{\frac{1}{2}}}}{2-\frac{1}{k^{\frac{2}{3}}}+1} \overset{\lim_{x\to\infty} \frac{1}{x^n}=0, n>0}{=} \frac{1}{2}$$

$$\lim_{k\to\infty} \left(\frac{2}{3}\right)^k \overset{\lim_{x\to\infty} n^x=0, n\in[0,1)}{=} 0$$

$$\therefore \lim_{k\to\infty} k\sin\left(\frac{1}{k}\right) = \lim_{k\to\infty} \frac{\sin\left(\frac{1}{k}\right)}{\frac{1}{k}} \overset{l=\frac{1}{k}}{=} \lim_{l\to 0} \frac{\sin(l)}{l} = 1$$

b) $\lim_{k\to\infty} \left(k\left(e^{\frac{1}{k}}-1\right), \left(1+\frac{2}{k+3}\right)^k\right)$

$$\lim_{k\to\infty} k\left(e^{\frac{1}{k}}-1\right) = \lim_{k\to\infty} \frac{e^{\frac{1}{k}}-1}{\frac{1}{k}} \overset{l=\frac{1}{k}}{=} \lim_{l\to 0} \frac{e^l-1}{l} = 1$$

$$\lim_{k\to\infty} \left(1+\frac{2}{k+3}\right)^k = \lim_{k\to\infty} \left(\left(1+\frac{2}{k+3}\right)^{\frac{k+3}{2}}\right)^{\cancel{\frac{2k}{k+3}} \nearrow 2} = e^2$$

$$\therefore \lim_{k\to\infty} \left(k\left(e^{\frac{1}{k}}-1\right), \left(1+\frac{2}{k+3}\right)^k\right) = (1, e^2)$$

c) $\lim\limits_{k\to\infty}\left(\sqrt[3]{k^2+k}-\sqrt[3]{k^2}, \left(\frac{2k-5}{2k-2}\right)^{4k^2}\right)$

$$\lim_{k\to\infty}\sqrt[3]{k^2+k}-\sqrt[3]{k^2}=\lim_{k\to\infty}\left(\sqrt[3]{k^2+k}-\sqrt[3]{k^2}\right)\frac{\sqrt[3]{k^2+k}+\sqrt[3]{k^4+k^3}+\sqrt[3]{k^2}}{\sqrt[3]{k^2+k}+\sqrt[3]{k^4+k^3}+\sqrt[3]{k^2}}$$

$$\lim_{k\to\infty}\frac{k^2+k-k^2}{\sqrt[3]{k^2+k}+\sqrt[3]{k^4+k^3}+\sqrt[3]{k^2}}=\lim_{k\to\infty}\frac{\frac{1}{k^{\frac{1}{3}}}}{\sqrt[3]{\frac{1}{k^2}+\frac{1}{k^3}}+\sqrt[3]{1+\frac{1}{k}}+\sqrt[3]{\frac{1}{k^2}}}\overset{\lim\limits_{x\to\infty}\frac{1}{x^n}=0,n>0}{=}0$$

$$\lim_{k\to\infty}\left(\frac{2k-5}{2k-2}\right)^{4k^2}=\lim_{k\to\infty}\left(\left(1-\frac{3}{2k-2}\right)^{\left(-\frac{2k-2}{3}\right)}\right)^{-\frac{12k^2}{2k-2}}=\lim_{k\to\infty}\frac{1}{\left(\left(1-\frac{3}{2k-2}\right)^{\left(-\frac{2k-2}{3}\right)}\right)^{\frac{12k}{2-\frac{2}{k}}}}=0$$

$$\therefore\lim_{k\to\infty}\left(\sqrt[3]{k^2+k}-\sqrt[3]{k^2}, \left(\frac{2k-5}{2k-2}\right)^{4k^2}\right)=(0,0)$$