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CS102 Calculus 3 Section G - Homework 3

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December 11, 2024

Problem 1 (20 points).

Let $r : [0, 2\pi] \to \mathbb{R}^3$ be the curve given by $r(t) = (2\cos t, 2\sin t, e^t)$.

a) Find the points on the curve, where the tangent line is perpendicular to the vector $\mathbf{a} = (\sqrt{3}, 1, 0)$.

$$r'(t) = (-2\sin t, 2\cos t, e^t), r'(t) \cdot \mathbf{a} = 0$$

$$-2\sqrt{3}\sin t + 2\cos t = 0 \implies \tan t = \frac{1}{\sqrt{3}} \implies t = \left\{\frac{pi}{6}, \frac{7\pi}{6}\right\}$$

$$A = \left(\sqrt{3}, 1, e^{\frac{\pi}{6}}\right), B = \left(-\sqrt{3}, -1, e^{\frac{7\pi}{6}}\right)$$

b) Find the parametric equations of these tangent lines.

$$r'\left(\frac{\pi}{6}\right) = \left(-1, \sqrt{3}, e^{\frac{\pi}{6}}\right) \text{ and } r'\left(\frac{7\pi}{6}\right) = \left(1, -\sqrt{3}, e^{\frac{7\pi}{6}}\right)$$

$$l_1 : \begin{bmatrix} \sqrt{3} \\ 1 \\ e^{\frac{\pi}{6}} \end{bmatrix} + p \begin{bmatrix} -1 \\ \sqrt{3} \\ e^{\frac{\pi}{6}} \end{bmatrix}, p \in \mathbb{R} \text{ and } l_2 : \begin{bmatrix} -\sqrt{3} \\ -1 \\ e^{\frac{7\pi}{6}} \end{bmatrix} + q \begin{bmatrix} 1 \\ -\sqrt{3} \\ e^{\frac{7\pi}{6}} \end{bmatrix}, q \in \mathbb{R}$$

Problem 2 (20 points).

Let $\gamma : [0, 1] \to \mathbb{R}^3$ be the space curve which satisfies

$$\gamma(t) = \left(t - \frac{1}{2}\right)^2 \langle t^2 \sin\left(\frac{1}{t}\right), \frac{\ln(1+t) - t}{t^2}, (t-1)\cos\left(\frac{1}{t-1}\right)\rangle$$

for all $0 < t < 1$

a) Find the start point and the end point of γ.

$$\lim_{t\to 0} \gamma(t_0) = \lim_{t\to 0} \left(t - \frac{1}{2}\right)^2 \langle t^2 \sin\left(\frac{1}{t}\right), \frac{\ln(1+t) - t}{t^2}, (t-1)\cos\left(\frac{1}{t-1}\right)\rangle$$

$$= \left(t - \frac{1}{2}\right)^2 \langle t\frac{\sin\left(\frac{1}{t}\right)}{\frac{1}{t}}, \frac{\frac{1}{1+t} - 1}{2t}, (t-1)\cos\left(\frac{1}{t-1}\right)\rangle = \langle 0, -\frac{1}{8}, -\frac{\cos(1)}{4}\rangle$$

$$\lim_{t\to 0}\gamma(t_1) = \lim_{t\to 1}\left(t-\frac{1}{2}\right)^2 \langle t^2\sin\left(\frac{1}{t}\right), \frac{\ln(1+t)-t}{t^2}, (t-1)\cos\left(\frac{1}{t-1}\right)\rangle$$

$$\left|\cos\left(\frac{1}{t-1}\right)\right| \le 1 \implies \left|(t-1)\cos\left(\frac{1}{t-1}\right)\right| \le |t-1| \to 0$$

$$= \lim_{t\to 1}\left(t-\frac{1}{2}\right)^2 \langle t^2\sin\left(\frac{1}{t}\right), \frac{\ln(1+t)-t}{t^2}, (t-1)\cos\left(\frac{1}{t-1}\right)\rangle = \langle\frac{\sin(1)}{4}, \ln(2)+1, 0\rangle$$

b) For each $t \in \left\{0, \frac{1}{2}, 1\right\}$determine if γ is differentiable at t and find $\gamma'(t)$ whenever it exists. Are the corresponding points on the curve regular ?

Since the function is not defined at $t=0$ and $t=1$, it is not continuous and is not differentiable at that points.

$$\gamma(t) = \langle\left(t-\frac{1}{2}\right)^2 t^2\sin\left(\frac{1}{t}\right), \left(t-\frac{1}{2}\right)^2\frac{\ln(1+t)-t}{t^2}, \left(t-\frac{1}{2}\right)^2 (t-1)\cos\left(\frac{1}{t-1}\right)\rangle$$

$$\gamma'\left(\frac{1}{2}\right) = \lim_{\Delta t\to 0}\frac{\gamma\left(\frac{1}{2}-\Delta t\right)-\gamma\left(\frac{1}{2}\right)}{\Delta t} = \lim_{\Delta t\to 0}\frac{\gamma\left(\frac{1}{2}-\Delta t\right)}{\Delta t}$$

$$\lim_{\Delta t\to 0}\frac{\langle\frac{1}{4}(\Delta t)^2\sin\left(\frac{1}{\frac{1}{2}-\Delta t}\right), (\Delta t)^2\frac{\ln(1.5-\Delta t)-0.5+\Delta t}{(0.5-\Delta t)^2}, (\Delta t)^2\left(-\frac{1}{2}-\Delta t\right)\cos\left(\frac{1}{-\frac{1}{2}-\Delta t}\right)\rangle}{\Delta t} = 0$$

$$\because \gamma'\left(\frac{1}{2}\right) = 0 \implies \text{ the points } t=\frac{1}{2} \text{ is not a regular point}$$

Problem 3 (40 points).

Find the length of the curve.

a) $r(t) = t\sqrt{2}\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}$, $0 \le t \le 1$.

$$L = \int_0^1 \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2}\,\mathrm{d}t = \int_0^1 \sqrt{2+e^{2t}+e^{-2t}}\,\mathrm{d}t = \int_0^1 \sqrt{2(1+\cosh(2t))}\,\mathrm{d}t$$

$$= \int_0^1 2\cosh(t))\,\mathrm{d}t = 2\left[\sinh t\right]_0^1 = \frac{e^2-1}{e}$$

b) $r(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + \ln(\cos(t))\mathbf{k}$, $0 \le t \le \frac{\pi}{4}$.

$$L = \int_0^{\frac{\pi}{4}} \sqrt{(-\sin t)^2 + (\cos t)^2 + (-\tan t)^2}\,\mathrm{d}t = \int_0^{\frac{\pi}{4}} \sqrt{\sin^2 t + \cos^2 t + \tan^2 t}\,\mathrm{d}t$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\cos t}\,\mathrm{d}t = \left[\frac{1}{2}\ln\left|\frac{1+\sin t}{1-\sin t}\right|\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\ln\left|\frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}}\right| = \frac{1}{2}\ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|$$

c) $r(t) = 12t\mathbf{i} + 8t^{\frac{3}{2}}\mathbf{j} + 3t^2\mathbf{k}$, $0 \le t \le 1$.

$$L = \int_0^1 \sqrt{(12)^2 + (12\sqrt{t})^2 + (6t)^2}\,\mathrm{d}t = 6\int_0^1 \sqrt{8+t^2}\,\mathrm{d}t \overset{t=\sqrt{8}\tan(\varphi)}{=} 12\sqrt{2}\int_0^{\frac{\pi}{4}} \frac{1}{\cos^3\varphi}\,\mathrm{d}\varphi$$

$$= 12\sqrt{2}\left[\frac{1}{2}\frac{\tan\varphi}{\cos\varphi} + \frac{1}{4}\ln\left|\frac{1+\sin\varphi}{1-\sin\varphi}\right|\right]_0^{\frac{\pi}{4}} = 12\sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{1}{4}\ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|\right)$$

d) $r(t) = \langle \frac{t^2}{2}, \frac{t^2}{2}, \frac{t^3}{3}, \frac{t^3}{3}, \frac{t^4}{4} \rangle$, $0 \le t \le a$.

$$L = \int_0^a \sqrt{2t^2 + 2t^4 + t^6}\,\mathrm{d}t = \int_0^a t\sqrt{t^4 + 2t^2 + 2}\,\mathrm{d}t \overset{u=t^2}{=} \frac{1}{2}\int_0^{a^2} \sqrt{(u+1)^2 + 1}\,\mathrm{d}u$$

$$\overset{v=u+1}{=} \frac{1}{2}\int_1^{a^2+1} \sqrt{v^2+1}\,\mathrm{d}v \overset{w=\sinh v}{=} \frac{1}{2}\int_{\sinh^{-1}(1)}^{\sinh^{-1}(a^2+1)} \cosh^2 w\,\mathrm{d}w$$

$$= \frac{1}{4}\int_{\sinh^{-1}(1)}^{\sinh^{-1}(a^2+1)} \cosh(2w) + 1\,\mathrm{d}w = \frac{1}{4}\left[\frac{\sinh(2w)}{2} + w\right]_{\sinh^{-1}(1)}^{\sinh^{-1}(a^2+1)}$$

$$= \frac{\sinh\bigl(2\sinh^{-1}(a^2+1)\bigr)}{8} + \frac{\sinh^{-1}(a^2+1)}{4} - \frac{\sinh\bigl(2\sinh^{-1}(1)\bigr)}{8} - \frac{\sinh^{-1}(1)}{4}$$

Problem 4 (20 points).

Suppose you start at the point $(0, 0, 3)$ and move 5 units along the curve $x = 3\sin(t), y = 4t, z = 3\cos(t)$. What are your possible locations now?

$$t = 0 \implies (0, 0, 3) \text{ is on the curve. Take } a \in \mathbb{R}$$

$$L = \int_0^a \sqrt{9\cos^2 t + 16 + 9\sin^2 t}\,\mathrm{d}t = \int_0^a 5\,\mathrm{d}t = 5a = 5 \implies a = 1$$

$$\therefore A = (3\sin(1), 4, 3\cos(1) \text{ is 5 units along the curve from } (0, 0, 3)$$

Also take from the other direction:

$$L = \int_a^0 \sqrt{9\cos^2 t + 16 + 9\sin^2 t}\,\mathrm{d}t = \int_a^0 5\,\mathrm{d}t = -5a = 5 \implies a = -1$$

$$\therefore A = (3\sin(-1), -4, 3\cos(-1) \text{ is 5 units along the curve from } (0, 0, 3)$$

Some magic integrals used:

$$I =: \int \frac{1}{\cos^3 x}dx = \int \frac{1}{\cos x} d(\tan x) \overset{\text{IbP}}{=} \frac{\tan x}{\cos x} - \int \tan x \frac{\tan x}{\cos x} dx$$

$$= \frac{\tan x}{\cos x} - \int \frac{1}{\cos^3 x}dx + \int \frac{1}{\cos x}dx = \frac{\tan x}{\cos x} - I + \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|$$

$$I = \frac{1}{2}\frac{\tan x}{\cos x} + \frac{1}{4}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + c$$

$$\int \frac{1}{\cos x}dx = \int \frac{\cos x}{(1-\sin^2 x)}dx = \int \frac{1}{(1-\sin^2 x)} d\,(\sin x)$$

$$\overset{u=\sin x}{=} \int \frac{1}{1-u^2}du = \int \frac{1}{(1+u)(1-u)}du = \frac{1}{2}\int \frac{1}{(1+u)} + \frac{1}{(1-u)}du$$

$$\overset{\text{FTC}}{=} \frac{1}{2}\left[\ln|1+u| - \ln|1-u|\right] = \frac{1}{2}\left(\ln\left|\frac{1+\sin x}{1-\sin x}\right|\right) + c$$