Digital version

CS102 Calculus 3 Section G - Homework 4

Mher Saribekyan A09210183

September 24, 2024

Problem 1 (20 points).

Find the limit, if it exists, or show that the limit does not exist.

a) $\lim\limits_{(x,y)\to(1,0)} \dfrac{\ln(x+e^y)}{\sqrt{x^2+y^2}}$

$$\lim_{(x,y)\to(1,0)} \frac{\ln(x+e^y)}{\sqrt{x^2+y^2}} = \frac{\ln\left(1+e^0\right)}{\sqrt{1^2+0^2}} = \ln(2)$$

b) $\lim\limits_{(x,y)\to(0,0)} \left(x^2+y^2\right)^{|x|}$

Take $x = r\cos\varphi$ and $y = r\sin\varphi$

$$\lim_{(x,y)\to(0,0)} \left(x^2+y^2\right)^{|x|} = \lim_{(r,\varphi)\to(0,0)} 1^{|r\cos\varphi|} = 1$$

c) $\lim\limits_{(x,y)\to(0,1)} \dfrac{\ln\left(y^2-2y+2\right)\sin^2(x)}{x^4+(y-1)^4}$

$$\lim_{(x,y)\to(0,1)} \frac{\ln\left(y^2-2y+2\right)\sin^2(x)}{x^4+(y-1)^4} \overset{x=0}{=} \lim_{y\to1} \frac{\ln\left(y^2-2y+2\right)\cdot 0}{(y-1)^4} = 0$$

$$\lim_{(x,y)\to(0,1)} \frac{\ln\left(y^2-2y+2\right)\sin^2(x)}{x^4+(y-1)^4} \overset{y=x+1}{=} \lim_{x\to0} \frac{\ln\left(x^2+1\right)}{2x^2}\cdot\left(\frac{\sin x}{x}\right)^2 = \frac{1}{2}$$

$\therefore$ the limit does not exists

d) $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xy+yz^2+xz^2}{x^2+y^2+z^4}$

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^4} \overset{\substack{x=z^2\\y=z^2}}{=} \lim_{z\to0} \frac{z^4+z^4+z^4}{z^4+z^4+z^4} = 1$$

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^4} \overset{\substack{x=z\\y=z}}{=} \lim_{z\to0} \frac{z^2+2z^3}{2z^2+z^4} = \lim_{z\to0} \frac{1+2z}{2+z^2} = \frac{1}{2}$$

$\therefore$ the limit does not exist

Problem 2 (20 points).

Study the continuity of the following functions

a) $f(x,y) = \operatorname{sgn}(y^2 - x^3)$

Function is not continuous, as at points where $y^2 = x^3$ there is a discontinuity.

$$\lim_{(x,y)\to(0,0^+)} f(x,y) = 1, \lim_{(x,y)\to(0,0^-)} f(x,y) = -1, f(0,0) = 0$$

b) $f(x,y) = [x+y]$, where $[a]$ is the integer part of a.

$$\lim_{(x,y)\to(0,1^-)} [x+y] = 0, \lim_{(x,y)\to(0,1^+)} [x+y] = 1 \implies \text{ not continuous}$$

c) $f(x,y) = x\sin\left(\frac{1}{y}\right) + y\sin\left(\frac{1}{x}\right)$, $f(x,0) = f(0,y) = 0$

$$\left|\sin\left(\frac{1}{x}\right)\right| \leq 1 \implies \left|y\sin\left(\frac{1}{x}\right)\right| \leq y \text{ and } \left|\sin\left(\frac{1}{y}\right)\right| \leq 1 \implies \left|x\sin\left(\frac{1}{y}\right)\right| \leq x$$

$$\lim_{(x,y)\to(0,0)} f(x,y) \overset{\text{squeeze}}{=} 0 = f(0,0) \text{ and no other discontinuity points } \implies f \text{ is continuous}$$

d) $f(x,y) = \begin{cases} \dfrac{\ln(1+|xy|)}{2x^2+y^2}, \text{ if } (x,y) \neq (0,0) \\ 0, \text{ if } (x,y) = (0,0) \end{cases}$

$$\text{Take } x = \frac{r}{\sqrt{2}}\cos\varphi \text{ and } y = r\sin\varphi$$

$$\lim_{(x,y)\to(0,0)} \frac{\ln(1+|xy|)}{2x^2+y^2} = \lim_{(r,\varphi)\to(0,0)} \frac{\ln\big(1+\left|r^2\sin 2\varphi\right|\big)}{r^2\sin 2\varphi} \cdot \sin 2\varphi = 0$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0) \text{ and no other discontinuity points } \implies f \text{ is continuous}$$

Problem 3 (30 points).

Find the general equation of the tangent plane to the graph of the given function at the specified point P whenever it exists.

a) $f(x,y) = 3y^2 - 2x^2 + x$, $P = (2,-1,-3)$

$$\frac{\partial f}{\partial x} = 1 - 4x \text{ and } \frac{\partial f}{\partial y} = 6y \text{ are continuous}$$

$$\text{General equation of a tangent plane: } z - z_0 = f'_x(x_0,y_0)\cdot(x-x_0) + f'_y(x_0,y_0)\cdot(y-y_0)$$

$$z + 3 = -7(x-2) - 6(y+1) \implies 7x + 6y + z = 5$$

b) $f(x,y) = x\sin(x+y)$, $P = (-1,1,0)$

$$\frac{\partial f}{\partial x} = \sin(x+y) + x\cos(x+y) \text{ and } \frac{\partial f}{\partial y} = x\cos(x+y) \text{ are continuous}$$

$$z - 0 = -1(x+1) - 1(y-1) \implies x + y + z = 0$$

c) $f(x,y) = \sqrt[3]{x^3+y^3}$, $P = (0,0,0)$

$$\frac{\partial f}{\partial x} = \frac{3x^2}{(x^3+y^3)^{\frac{2}{3}}} \text{ and } \frac{\partial f}{\partial y} = \frac{3y^2}{(x^3+y^3)^{\frac{2}{3}}} \text{ are not continuous at point } P$$

$$\therefore \text{ the function does not have a tangent plane at point } P$$

Problem 4 (30 points).

For the following function f compute the partial derivatives of f at $P = (0,0)$ whenever they exist. Is f differentiable at P?

a) $f(x,y) = \sqrt{x^2+y^2}$

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(0,0)} = \lim_{\Delta x \to 0} \frac{\sqrt{(0+\Delta x)^2 + 0^2} - 0}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(0,0)} = \lim_{\Delta y \to 0} \frac{\sqrt{0^2 + (0+\Delta y)^2} - 0}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y}{\Delta y} = 1$$

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{f(0+\Delta x, 0+\Delta y) - f(0,0) - \Delta x - \Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} = \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\sqrt{\Delta x^2 + \Delta y^2} - (\Delta x + \Delta y)}{\sqrt{\Delta x^2 + \Delta y^2}}$$

$$= \lim_{(r,\varphi) \to (0,0)} 1 - \frac{r(\sin\varphi + \cos\varphi)}{r} = 0$$

$$\therefore f \text{ is differentiable at } P = (0,0)$$

b) $f(x,y) = \sqrt[3]{x^4+y^4}$

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(0,0)} = \lim_{\Delta x \to 0} \frac{\sqrt[3]{(0+\Delta x)^4 + 0^4} - 0}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x^{\frac{4}{3}}}{\Delta x} = 0$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(0,0)} = \lim_{\Delta y \to 0} \frac{\sqrt[3]{0^4 + (0+\Delta y)^4} - 0}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y^{\frac{4}{3}}}{\Delta y} = 0$$

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{f(0+\Delta x, 0+\Delta y) - f(0,0) - 0\Delta x - 0\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} = \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\sqrt[3]{\Delta x^4 + \Delta y^4}}{\sqrt{\Delta x^2 + \Delta y^2}}$$

$$= \lim_{(r,\varphi) \to (0,0)} \frac{r^{\frac{4}{3}} \sqrt[3]{\sin^4\varphi + \cos^4\varphi}}{r} = 0$$

$$\therefore f \text{ is differentiable at } P = (0,0)$$

c) $f(x,y) = \begin{cases} e^{-\dfrac{1}{x^2+y^2}}, \text{ if } (x,y) \neq (0,0) \\ 0, \text{ if } (x,y) = (0,0) \end{cases}$

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(0,0)} = \lim_{\Delta x \to 0} \frac{e^{-\frac{1}{\Delta x^2}}}{\Delta x} \overset{w=\frac{1}{\Delta x}}{=} \lim_{w \to \infty} \frac{w}{e^{w^2}} = 0$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(0,0)} = \lim_{\Delta y \to 0} \frac{e^{-\frac{1}{\Delta y^2}}}{\Delta y} \overset{w=\frac{1}{\Delta y}}{=} \lim_{w \to \infty} \frac{w}{e^{w^2}} = 0$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{f(0+\Delta x, 0+\Delta y) - f(0,0) - 0\Delta x - 0\Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} = \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{e^{-\frac{1}{\Delta x^2+\Delta y^2}}}{\sqrt{\Delta x^2+\Delta y^2}}$$

$$\overset{w=\frac{1}{\Delta x^2+\Delta y^2}}{=} \lim_{w\to\infty} \frac{e^{-w}}{\sqrt{\frac{1}{w}}} = \lim_{w\to\infty} \frac{\sqrt{w}}{e^{w}} = 0$$

$$\therefore f \text{ is differentiable at } P = (0,0)$$