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CS102 Calculus 3 Section G - Homework 5

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Problem 1 (10 points).

A model for the surface area of a human body is given by $S(w,h) = 0.1091w^{0.425}h^{0.725}$, where w is the weight (in pounds), h is the height (in inches), and S is measured in square feet. If the errors in measurement of w and h are at most 2%, use differentials to estimate the maximum percentage error in the calculated surface area.

$$S(1.02w_0, 1.02h_0) \approx S(w_0,h_0) + S'_w(w_0,h_0)\cdot(0.02h_0) + S'_h(w_0,h_0)\cdot(0.02h_0)$$

$$\Delta S \approx (0.425\cdot 0.1091w_0^{0.575-1}h_0^{0.725})\cdot(0.02w_0) + (0.725\cdot 0.1091w_0^{0.425}h_0^{0.725-1})\cdot(0.02h_0)$$

$$\frac{\Delta S}{S} \approx 0.02\cdot(0.425+0.725) = 2.3\%$$

Problem 2 (20 points).

Find the indicated partial derivatives

a) $f(x,y) = x^4y^2 - x^3y$; f_{xxx}, f_{yyy}

$$f_{xxx} = 24xy^2 - 6y \text{ and } f_{xyx} = 24x^2y - 6x$$

b) $u = e^{r\theta}\sin(\theta)$; $\dfrac{\partial^3 u}{\partial r^2 \partial\theta}$

$$\frac{\partial^3 u}{\partial r^2\partial\theta} = \frac{\partial^2}{\partial r^2}\Big(re^{r\theta}\sin(\theta) + e^{r\theta}\cos(\theta)\Big) = \Big(e^{r\theta} + \theta e^{r\theta} + r\theta^2 e^{r\theta}\Big)\sin(\theta) + \theta^2 e^{r\theta}\cos(\theta)$$

c) $w = \dfrac{x}{y+2z}$; $\dfrac{\partial^3 w}{\partial z\partial y\partial x}$, $\dfrac{\partial^3 w}{\partial x^2\partial y}$

$$\frac{\partial^3 w}{\partial z\partial y\partial x} = \frac{4}{(y+2z)^3} \text{ and } \frac{\partial^3 w}{\partial x^2\partial y} = 0$$

d) $u = x^ay^bz^c$; $\dfrac{\partial^6 u}{\partial x\partial y^2\partial z^3}$

$$\frac{\partial^6 u}{\partial x\partial y^2\partial z^3} = ab(b-2)c(c-2)(c-3)x^{a-1}y^{b-2}z^{c-3}$$

Problem 3 (20 points).

Verify that the given function is a solution to the specified differential equation

a) $z = \ln(e^x + e^y)$; $\dfrac{\partial^2 z}{\partial x^2}\dfrac{\partial^2 z}{\partial y^2} - \left(\dfrac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$

$$\frac{\partial z}{\partial x} = \frac{e^x}{e^x + e^y} = 1 - \frac{e^y}{e^x + e^y} \text{ and } \frac{\partial z}{\partial y} = \frac{e^y}{e^x + e^y} = 1 - \frac{e^x}{e^x + e^y}$$

$$\left(\frac{e^x e^y}{(e^x + e^y)^2}\right)\left(\frac{e^y e^x}{(e^y + e^x)^2}\right) - \left(-\frac{e^x e^y}{(e^x + e^y)^2}\right)^2 = \frac{e^{2x}e^{2y}}{(e^x + e^y)^4} - \frac{e^{2x}e^{2y}}{(e^x + e^y)^4} = 0$$

b) $u = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$; $u_{xx} + u_{yy} + u_{zz} = 0$

$$u_x = -\frac{x}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \implies u_{xx} = \frac{3x^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} - \frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

$$u_y = -\frac{y}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \implies u_{yy} = \frac{3y^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} - \frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

$$u_z = -\frac{z}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} \implies u_{zz} = \frac{3z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} - \frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}$$

$$u_{xx} + u_{yy} + u_{zz} = \frac{3x^2 + 3y^2 + 3z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} - 3\frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} = 0$$

Problem 4 (20 points).

Use the Chain Rule to find the indicated partial derivatives

a) $z = x^4 + x^2 y$, $x = s + 2t - u$, $y = stu^2$; $\dfrac{\partial z}{\partial s}, \dfrac{\partial z}{\partial t}, \dfrac{\partial z}{\partial u}$ when $s = 4$, $t = 2$, $u = 1$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (4x^3 + 2xy)(1) + (x^2)(tu^2)$$

$$\left.\frac{\partial z}{\partial s}\right|_{(s,t,u)=(4,2,1)} = 4 \cdot 7^3 + 2 \cdot 8 \cdot 7 + 7^2 \cdot 2 = 1582$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (4x^3 + 2xy)(2) + (x^2)(su^2)$$

$$\left.\frac{\partial z}{\partial t}\right|_{(s,t,u)=(4,2,1)} = 2(4 \cdot 7^3 + 2 \cdot 8 \cdot 7) + (7^2)(4) = 3164$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (4x^3 + 2xy)(-1) + (x^2)(2stu)$$

$$\left.\frac{\partial z}{\partial u}\right|_{(s,t,u)=(4,2,1)} = -(4 \cdot 7^3 + 2 \cdot 8 \cdot 7) + (7^2)(64) = 1652$$

b) $w = xy + yz + zx$, $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = r\theta$; $\dfrac{\partial w}{\partial r}, \dfrac{\partial w}{\partial \theta}$ when $r = 2$, $\theta = \dfrac{\pi}{2}$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} = (y+z)(\cos\theta) + (x+z)(\sin\theta) + (x+y)(\theta)$$

$$\left.\frac{\partial w}{\partial r}\right|_{(r,\theta)=(2,\pi/2)} = 0 + (0+\pi) + \frac{\pi}{2}(2) = 2\pi$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \partial w \partial z\frac{\partial z}{\partial \theta} = (y+z)(-r\sin\theta) + (x+z)(r\cos\theta) + (x+y)(r)$$

$$\left.\frac{\partial w}{\partial \theta}\right|_{(r,\theta)=(2,\pi/2)} = (2+\pi)(-2) + (0+\pi)(0) + (0+2)(2) = -2\pi$$

Problem 5 (20 points).

For the following function f evaluate the gradient of f at the given point P and compute the directional derivative of f at P in the direction of the specified vector v. What is the maximum rate of change of f at P in the direction in which it occurs?

a) $f(x, y) = e^x \sin(y)$; $P = \left(0, \frac{\pi}{3}\right)$, $\mathbf{v} = \{-6, 8\}$

$$\nabla f(x,y) = (e^x\sin(y), e^x\cos(y)) \implies \nabla f\left(0, \frac{\pi}{3}\right) = \left\{\frac{\sqrt{3}}{2}, \frac{1}{2}\right\}$$

$$\text{Take unit vector } \mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\{-6,8\}}{\sqrt{36+64}} = \left\{-\frac{3}{5}, \frac{4}{5}\right\} \implies \nabla_{\mathbf{v}} f\left(0, \frac{\pi}{3}\right) = \frac{4-3\sqrt{3}}{10}$$

b) $f(x, y) = x^4 - x^2y^3$; $P = (2, 1)$, $v = \{1, 3\}$

$$\nabla f(x,y) = (4x^3 - 2xy^3, -3x^2y^2) \implies \nabla f(1,3) = \{-50, -27\}$$

$$\text{Take unit vector } \mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\{1,3\}}{\sqrt{1+9}} = \left\{\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right\} \implies \nabla_{\mathbf{v}} f(1,3) = -\frac{131}{\sqrt{10}}$$

c) $f(x, y, z) = xe^y + ye^z + ze^x$; $P = (0, 0, 0)$, $v = \{5, 1, -2\}$

$$\nabla f(x,y,z) = (e^y + ye^z + ze^x, xe^y + e^z + ze^x, xe^y + ye^z + e^x) \implies \nabla f(0,0,0) = \{1,1,1\}$$

$$\text{Take unit vector } \mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\{5,1,-2\}}{\sqrt{25+1+4}} = \left\{\frac{5}{\sqrt{30}}, \frac{1}{\sqrt{30}}, -\frac{2}{\sqrt{30}}\right\} \implies \nabla_{\mathbf{v}} f(0,0,0) = \frac{4}{\sqrt{30}}$$

d) $f(x, y, z) = \ln(3x + 6y + 9z)$; $P = (1, 1, 1)$, $v = \{4, 12, 6\}$

$$\nabla f(x,y,z) = \left(\frac{3}{3x+6y+9z}, \frac{6}{3x+6y+9z}, \frac{9}{3x+6y+9z}\right) \implies \nabla f(1,1,1) = \left\{\frac{1}{6}, \frac{2}{6}, \frac{3}{6}\right\}$$

$$\text{Take unit vector } \mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\{4,12,6\}}{\sqrt{16+144+36}} = \left\{\frac{2}{7}, \frac{6}{7}, \frac{3}{7}\right\} \implies \nabla_{\mathbf{v}} f(1,1,1) = \frac{11}{7}$$

Problem 6 (10 points).

Assume that $f = f(x, y)$ has continuous second-order partial derivatives. If $x = e^s \cos(t)$ and $y = e^s \sin(t)$, show that

$$f_{xx} + f_{yy} = e^{-2s}(f_{ss} + f_{tt})$$

$$f_s = f_x x_s + f_y y_s = f_x x + f_y y$$

$$f_{ss} = f_{sx} x_s + f_{sy} y_s = (f_{sx})x + (f_{sy})y = (f_{xx}x + f_x + f_{yx}y + f_y y_x)x + (f_{xy}x + f_x x_y + f_{yy}y + f_y)y$$

$$f_t = f_x x_t + f_y y_t = f_y x - f_x y$$

$$f_{tt} = f_{tx} x_t + f_{ty} y_t = (f_{yy}x + f_y x_y - f_{xy}y - f_x)x - (f_{yx}x + f_y - f_{xx}y - f_x y_x)y$$

$$f_{ss} = f_{xx}x^2 + f_x x + f_{yx}xy + f_y x y_x + f_{xy}xy + f_x x_y y + f_{yy}y^2 + f_y y$$

$$f_{tt} = f_{yy}x^2 + f_y x x_y - f_{xy}xy - f_x x - f_{yx}xy - f_y y + f_{xx}y^2 + f_x y_x y$$

$$\because x_s = y, y_s = x \text{ and } x_t = -y, y_t = x \implies f_{ss} + f_{tt} = f_{xx}x^2 + f_{xx}y^2 + f_{yy}x^2 + f_y y^2 = e^{2s}(f_{ss} + f_{tt})$$

$$\therefore f_{xx} + f_{yy} = e^{-2s}(f_{ss} + f_{tt})$$