Digital version

CS102 Calculus 3 Section G - Homework 6

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Problem 1 (20 points).

Let$A = \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix}$ be the given positive semidefinite matrix. For the specified point P give an example of a function $f = f(x, y)$ satisfying the following condition or prove that it does not exist.

a) $f(P) = 1$, $H_f(P) = A$ and f has a local minimum at P; $P = (0, 0)$

$$f(x, y) = 2x^2 + 6xy + 4.5y^2 + x^4 + 1$$

b) $f(P) = 2$, $H_f(P) = -A$ and f has a local maximum at P; $P = (1, 0)$

$$f(x, y) = -2(x-1)^2 - 6(x-1)y - 4.5y^2 - (x-1)^4 + 2$$

c) $f(P) = 0$, $H_f(P) = A$ and P is a saddle point of f; $P = (0, 0)$

$$f(x, y) = 2x^2 + 6xy + 4.5y^2 + x^3$$

d) $H_f(P) = A$ and f has a local maximum at P; $P = (0, 0)$

Since the Hessian matrix is positive semidefinite, the point P can either be a saddle point or a local minimum. Hence, there does not exist such a function, where P is a local maximum.

Problem 2 (20 points).

Identify and classify the critical points of the given function.

a) $f(x, y) = x^2 + xy + y^2 + y$

$$\begin{cases} \frac{\partial f}{\partial x} = 2x + y = 0 \\ \frac{\partial f}{\partial y} = x + 2y + 1 = 0 \end{cases} \implies 3x - 1 = 0 \implies (x, y) = \left(\frac{1}{3}, -\frac{2}{3}\right) \implies P_1 = \left(\frac{1}{3}, -\frac{2}{3}, -\frac{1}{3}\right)$$

$$H_f = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \text{ is positive definite } \implies P_1 \text{ is a local minimum}$$

b) $f(x,y) = x^3 - 12xy + 8y^3$

$$\begin{cases} \frac{\partial f}{\partial x} = 3x^2 - 12y = 0 \\ \frac{\partial f}{\partial y} = 24y^2 - 12x = 0 \end{cases} \implies \begin{cases} x^2 = 4y \\ 2y^2 = x \end{cases} \implies y(y^3-1) = 0 \implies \begin{cases} P_1 = (0,0,0) \\ P_2 = (2,1,-8) \end{cases}$$

$$H_f(P_1) = \begin{bmatrix} 0 & -12 \\ -12 & 0 \end{bmatrix} \text{ is indefinite} \implies P_1 \text{ is a saddle point}$$

$$H_f(P_2) = \begin{bmatrix} 12 & -12 \\ -12 & 48 \end{bmatrix} \text{ is positive definite} \implies P_2 \text{ is a local minimum}$$

c) $f(x,y) = (x^2+y^2)e^{y^2-x^2}$

$$\begin{cases} \frac{\partial f}{\partial x} = 2xe^{y^2-x^2} - 2x(x^2+y^2)e^{y^2-x^2} = 0 \\ \frac{\partial f}{\partial y} = 2ye^{y^2-x^2} + 2y(x^2+y^2)e^{y^2-x^2} = 0 \end{cases} \implies \begin{cases} 2x = 2x(x^2+y^2) \\ 2y = -2y(x^2+y^2) \end{cases} \implies \begin{cases} P_1 = (0,0,0) \\ P_2 = (1,0,1/e) \\ P_3 = (-1,0,1/e) \end{cases}$$

$$\frac{\partial^2 f}{\partial x^2} = (2-4x-2y^2)e^{y^2-x^2} - 2x(2x-2x^3-2xy^2)e^{y^2-x^2}$$

$$\frac{\partial^2 f}{\partial y \partial x} = (-4xy)e^{y^2-x^2} + 2y(2x-2x^3-2xy^2)e^{y^2-x^2}$$

$$\frac{\partial^2 f}{\partial y^2} = (2+2x^2+4y^2)e^{y^2-x^2} + 2y(2y+2yx^2+2y^3)e^{y^2-x^2}$$

$$H_f(P_1) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, H_f(P_2) = \begin{bmatrix} -\frac{2}{e} & 0 \\ 0 & \frac{4}{e} \end{bmatrix}, H_f(P_3) = \begin{bmatrix} -\frac{6}{e} & 0 \\ 0 & \frac{4}{e} \end{bmatrix}$$

$$\therefore P_1 \text{ is a local minimum}, P_2, P_3 \text{ are saddle points}$$

d) $f(x,y) = x^2ye^{-x^2-y^2}$

$$\begin{cases} \frac{\partial f}{\partial x} = (1-x^2)2xye^{-x^2-y^2} = 0 \\ \frac{\partial f}{\partial y} = (1-2y^2)x^2e^{-x^2-y^2} = 0 \end{cases} \implies \begin{cases} (1-x^2)2xy = 0 \\ (1-2y^2)x^2 = 0 \end{cases} \implies \begin{cases} P_0 = (0,0,0) \\ P_1 = \left(1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}e^{1.5}}\right) \\ P_2 = \left(1, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}e^{1.5}}\right) \\ P_3 = \left(-1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}e^{1.5}}\right) \\ P_4 = \left(-1, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}e^{1.5}}\right) \end{cases}$$

$$\therefore \begin{cases} P_0 \text{ is a saddle point} \\ P_1 \text{ is a local maximum} \\ P_2 \text{ is a local minimum} \\ P_3 \text{ is a local maxmimum} \\ P_4 \text{ is a local minimum} \end{cases}$$

Problem 3 (20 points).

Find the absolute maximum and minimum values of the function f on the set D.

a) $f(x,y) = x^2 + y^2 - 2x$, D is the closed triangular region with vertices $(2,0)$, $(0,2)$, and $(0,-2)$

$$\begin{cases} \frac{\partial f}{\partial x} = 2x - 2 = 0 \\ \frac{\partial f}{\partial y} = 2y = 0 \end{cases} \implies (x,y) = (1,0) \text{ is inside the triangle}$$

$$\text{The triangle is bounded by } \begin{cases} y = 2 - x, x \in [0,2] \\ x = 0, y \in [-2,2] \\ y = x - 2, x \in [0,2] \end{cases}$$

$$\begin{cases} f(x, 2-x) = x^2 + (2-x)^2 - 2x, x \in [0,2] \\ f(0,y) = y^2, y \in [-2,2] \\ f(x, x-2) = x^2 + (x-2)^2 - 2x, x \in [0,2] \end{cases} \implies \begin{cases} f(x, 2-x) = 2x^2 - 6x + 4, x \in [0,2] \\ f(0,y) = y^2, y \in [-2,2] \\ f(x, x-2) = 2x^2 - 6x + 4, x \in [0,2] \end{cases}$$

$$\implies \begin{cases} f(2,0) = 0 \\ f(0,2) = 4 \text{ is absolute maximum} \\ f(0,-2) = 4 \text{ is absolute maximum} \\ f(0,0) = 0 \\ f(1.5, 0.5) = -0.5 \\ f(1.5, -0.5) = -0.5 \\ f(1,0) = -1 \text{ is absolute minimum} \end{cases}$$

b) $f(x,y) = xy^2$, $D = \{(x,y) | x \geq 0, y \geq 0, x^2 + y^2 \leq 3\}$

$$\begin{cases} \frac{\partial f}{\partial x} = y^2 = 0 \\ \frac{\partial f}{\partial y} = 2xy = 0 \end{cases} \implies y = 0 \text{ or } x = 0$$

$$\text{The quarter-circle is bounded by } \begin{cases} x^2 + y^2 = 3, x, y \geq 0 \\ y = 0, x \in [0,3] \\ x = 0, y \in [0,3] \end{cases}$$

$$\begin{cases} f(x, \sqrt{3 - x^2}) = x(3 - x^2), x \in [0,3] \\ f(x,0) = 0, x \in [0,3] \\ f(0,y) = 0, y \in [0,3] \end{cases} \implies \begin{cases} f(1, \sqrt{2}) = 2 \text{ is absolute maximum} \\ f(x,0) = 0, x \in [0,3] \text{ is absolute minimum} \\ f(0,y) = 0, y \in [0,3] \text{ is absolute minimum} \end{cases}$$

Problem 4 (20 points).

Use Lagrange multipliers to find the maximum and minimum values of the function subject to the given constraint

a) $f(x,y,z) = 2x + 2y + z$; $x^2 + y^2 + z^2 = 9$

$$g(x,y,z) := x^2 + y^2 + z^2 - 9$$

$$\begin{cases} 2 = \lambda 2x \\ 2 = \lambda 2y \\ 1 = \lambda 2z \\ x^2 + y^2 + z^2 - 9 = 0 \end{cases} \implies \begin{cases} \lambda = \dfrac{1}{x} = \dfrac{1}{y} = \dfrac{1}{2z} \\ x^2 + y^2 + z^2 = 9 \end{cases} \implies 9z^2 = 9 \implies \begin{cases} f_{max} = f(2,2,1) = 9 \\ f_{min} = f(-2,-2,-1) = -9 \end{cases}$$

b) $f(x, y, z) = xyz$; $x^2 + 2y^2 + 3z^2 = 6$

$$g(x, y, z) := x^2 + 2y^2 + 3z^2 - 6$$

$$\begin{cases} yz = \lambda 2x \\ xz = \lambda 4y \\ xy = \lambda 6z \\ x^2 + 2y^2 + 3z^2 - 6 = 0 \end{cases} \implies \begin{cases} 2y^2 = x^2 \\ 3z^2 = 2y^2 \\ 3z^2 = x^2 \\ 3x^2 = 6 \end{cases} \implies \begin{cases} f_{max} = f\left(\sqrt{2}, 1, \sqrt{\frac{2}{3}}\right) = \dfrac{2}{\sqrt{3}} \\ f_{min} = f\left(-\sqrt{2}, 1, \sqrt{\frac{2}{3}}\right) = -\dfrac{2}{\sqrt{3}} \end{cases}$$

Problem 5 (20 points).

a) Find the points on the surface $y^2 = 9 + xz$ that are closest to the origin.

$$d^2 = x^2 + y^2 + z^2 \text{ and } g(x, y, z) := xz - y^2 + 9 = 0$$

$$\begin{cases} 2x = \lambda z \\ 2y = \lambda(-2y) \\ 2z = \lambda x \\ xz - y^2 + 9 = 0 \end{cases} \implies \begin{cases} \lambda = -1 \implies x, z = 0, y = \pm 3 \\ \lambda = 0, y = 0 \implies x = 3, z = -3 \text{ or } x = -3, z = 3 \end{cases}$$

$\therefore$ the points closest to the origin are $(0, 3, 0)$ and $(0, -3, 0)$, with $d = 3$

b) Find the dimensions of a rectangular box of maximum volume if the total surface area is given as $64cm^2$.

Denote x, y and z the dimensions of the box. The volume $V(x, y, z) = xyz$, the surface area, $S(x, y, z) = 2(xy + xz + yz) = 64$. Denote $g(x, y, z) := xy + xz + yz - 32 = 0$.

$$\begin{cases} yz = \lambda(y + z) \\ xz = \lambda(x + z) \\ xy = \lambda(x + y) \\ xy + xz + yz - 32 = 0 \end{cases} \implies \begin{cases} x = y = z \\ 3x^2 = 32 \end{cases} \implies x = y = z = \sqrt{\frac{32}{3}} \implies V = \frac{128\sqrt{6}}{9} cm^3$$