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# CS102 Calculus 3 Section G - Homework 7

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## Problem 1 (20 points).

Use Fubini's theorem to calculate the double integrals

a) $\int\int_{[0,\pi/2]\times[0,\pi/2]} \sin(x+y)\,\mathrm{d}x\,\mathrm{d}y$

$$\int\int_{[0,\pi/2]\times[0,\pi/2]} \sin(x+y)\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin(x+y)\,\mathrm{d}y\,\mathrm{d}x$$
$$= \int_0^{\frac{\pi}{2}} \cos(x)+\sin(x)\,\mathrm{d}x = 2$$

b) $\int\int_{[0,1]\times[-3,3]} \frac{xy^2}{x^2+1}\,\mathrm{d}x\,\mathrm{d}y$

$$\int\int_{[0,1]\times[-3,3]} \frac{xy^2}{x^2+1}\,\mathrm{d}x\,\mathrm{d}y = \int_0^1\int_{-3}^3 \frac{xy^2}{x^2+1}\,\mathrm{d}y\,\mathrm{d}x$$
$$= \int_0^1 \frac{18x}{x^2+1}\,\mathrm{d}x = \int_0^1 \frac{9}{x^2+1}\,\mathrm{d}(x^2+1) \overset{u=x^2+1}{=} \int_1^2 \frac{9}{u}\,\mathrm{d}u = 9\ln(2)$$

## Problem 2 (20 points).

Evaluate the double integral.

a) $\int\int_D x\cos(y)\,\mathrm{d}A$, $D$ is bounded by $y=0$, $y=x^2$, $x=1$

$$\int_0^1\int_0^{x^2} x\cos(y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 x\sin\left(x^2\right)\mathrm{d}x = \frac{1-\cos(1)}{2}$$

b) $\int\int_D y^2\,\mathrm{d}A$, $D$ is the triangular region with vertices $(0,1)$, $(1,2)$, $(4,1)$

$$\int_1^2\int_{y-1}^{7-3y} y^2\,\mathrm{d}x\,\mathrm{d}y = \int_1^2 8y^2-4y^3\,\mathrm{d}y = \frac{11}{3}$$

c) $\displaystyle\iint_D (2x-y)\,\mathrm{d}A$, $D$ is bounded by the circle of radius 2 centered at the origin

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2x-y\,\mathrm{d}y\,\mathrm{d}x = \int_{-2}^{2} 4x\sqrt{4-x^2}\,\mathrm{d}x = 0$$

d) $\displaystyle\iint_D y\,\mathrm{d}A$, $D$ is the region bounded by $x=-1$, $y=-1$, $y=(x+1)^2$, $x=y-y^3$

$$\int_{-1}^{0}\int_{-1}^{y-y^3} y\,\mathrm{d}x\,\mathrm{d}y + \int_{-1}^{0}\int_{0}^{(x+1)^2} y\,\mathrm{d}y\,\mathrm{d}x = \int_{-1}^{0} y^2-y^4+y\,\mathrm{d}y + \int_{-1}^{0}\frac{(x+1)^4}{2}\,\mathrm{d}x$$

$$= \frac{1}{3}-\frac{1}{5}-\frac{1}{2}+\frac{1}{10} = -\frac{4}{15}$$

## Problem 3 (20 points).

Find the volume of the solid

a) Under the surface $z=xy$ and above the triangle with vertices $(1,1)$, $(4,1)$, $(1,2)$.

$$V=\int_1^2\int_1^{7-3y} xy\,\mathrm{d}x\,\mathrm{d}y = \int_1^2 24y-21y^2+\frac{9}{2}y^3\,\mathrm{d}y = (48-56+18)-\left(12-7+\frac{9}{8}\right)=\frac{31}{8}$$

b) Bounded by the coordinate planes and the plane $3x+2y+z=6$.

$$V=\int_0^2\int_0^{3-\frac{3}{2}x} 6-3x-2y\,\mathrm{d}y\,\mathrm{d}x = \int_0^2 9-9x+\frac{9}{4}x^2\,\mathrm{d}x = 18-18+6=6$$

## Problem 4 (20 points).

Evaluate the integral by reversing the order of integration.

a) $\displaystyle\int_0^1\int_{3y}^3 e^{x^2}\,\mathrm{d}x\,\mathrm{d}y$

$$\int_0^1\int_{3y}^3 e^{x^2}\,\mathrm{d}x\,\mathrm{d}y = \int_0^3\int_0^{\frac{x}{3}} e^{x^2}\,\mathrm{d}y\,\mathrm{d}x = \frac{1}{6}\int_0^3 2xe^{x^2}\,\mathrm{d}x = \frac{e^9-1}{6}$$

b) $\displaystyle\int_0^1\int_{\arcsin(y)}^{\frac{\pi}{2}} \cos(x)\sqrt{1+\cos^2(x)}\,\mathrm{d}x\,\mathrm{d}y$

$$\int_0^1\int_{\arcsin(y)}^{\frac{\pi}{2}} \cos(x)\sqrt{1+\cos^2(x)}\,\mathrm{d}x\,\mathrm{d}y = \int_0^{\frac{\pi}{2}}\int_0^{\sin(x)} \cos(x)\sqrt{1+\cos^2(x)}\,\mathrm{d}y\,\mathrm{d}x$$

$$= -\int_0^{\frac{\pi}{2}} \cos(x)\sqrt{1+\cos^2(x)}\,\mathrm{d}\cos(x) = \frac{1}{2}\int_0^1 2u\sqrt{1+u^2}\,\mathrm{d}u = \frac{2\sqrt{2}-1}{3}$$

## Problem 5 (20 points).

Evaluate the given integral by changing to polar coordinates.

a) $\displaystyle\iint_D x^2 y \,\mathrm{d}A$, where $D$ is the top half of the disk with center the origin and radius 5

$$\int_0^{\pi}\int_0^5 r^3\cos^2\varphi\sin\varphi r\,\mathrm{d}r\,\mathrm{d}\varphi = -\int_0^{\pi} 5^4\cos^2\varphi\,\mathrm{d}\cos(\varphi) = \int_{-1}^{1} 5^4 u^2\,\mathrm{d}u = \frac{1250}{3}$$

b) $\displaystyle\iint_R \sin\left(x^2+y^2\right)\mathrm{d}A$, where $R$ is the region in the first quadrant between the circles centered at the origin of radii 1 and 3

$$\int_0^{\frac{\pi}{2}}\int_1^3 \sin\left(r^2\right) r\,\mathrm{d}r\,\mathrm{d}\varphi = \int_0^{\frac{\pi}{2}} \frac{1}{2}(\cos(1)-\cos(9))\,\mathrm{d}\varphi = \frac{\pi(\cos(1)-\cos(9))}{4}$$

c) $\displaystyle\iint_D e^{-x^2-y^2}\,\mathrm{d}A$, where $D$ is the region bounded by the semicircle $x=\sqrt{4-y^2}$ and the y-axis

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^2 e^{-r^2} r\,\mathrm{d}r\,\mathrm{d}\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-4}^{0}\frac{e^u}{2}\,\mathrm{d}u\,\mathrm{d}\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1-e^{-4}}{2}\,\mathrm{d}\varphi = \frac{\pi(e^4-1)}{2e^4}$$

d) $\displaystyle\iint_D x\,\mathrm{d}A$, where $D$ is the region in the first quadrant that lies between the circles $x^2+y^2=4$ and $x^2+y^2=2x$

$$x^2+y^2=2x \implies r=2\cos(\varphi)$$

$$\int_0^{\frac{\pi}{2}}\int_{2\cos(\varphi)}^{2} r\cos\varphi r\,\mathrm{d}r\,\mathrm{d}\varphi = \frac{8}{3}\int_0^{\frac{\pi}{2}}\cos\varphi-\cos^4\varphi\,\mathrm{d}\varphi$$

$$=\frac{8}{3}\int_0^{\frac{\pi}{2}}\cos\varphi-\frac{3}{8}-\frac{\cos(2\varphi)}{2}-\frac{\cos(4\varphi)}{8}\,\mathrm{d}\varphi = \frac{8}{3}\left(1-\frac{3\pi}{16}\right) = \frac{16-3\pi}{6}$$