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CS102 Calculus 3 Section G - Homework 8

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Problem 1 (20 points).

Evaluate the integral

a) $\iiint_T xyz \, \mathrm{d}V$, where T is the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(1,1,0)$ and $(1,0,1)$.

$$\int_0^1 \int_0^x \int_0^{x-y} xyz \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x = \frac{1}{2} \int_0^1 \int_0^x xy(x^2 - 2xy + y^2) \, \mathrm{d}y \, \mathrm{d}x = \frac{1}{2} \int_0^1 \int_0^x x^3 y - 2x^2 y^2 + xy^3 \, \mathrm{d}y \, \mathrm{d}x$$

$$= \frac{1}{2} \int_0^1 \frac{x^5}{2} - \frac{2x^5}{3} + \frac{x^5}{4} \, \mathrm{d}x = \frac{1}{24} \int_0^1 x^5 \, \mathrm{d}x = \frac{1}{144}$$

b) $\int_0^1 \int_0^{1-x} \int_0^{2-2z} \mathrm{d}y \, \mathrm{d}z \, \mathrm{d}x$

$$\int_0^1 \int_0^{1-x} \int_0^{2-2z} \mathrm{d}y \, \mathrm{d}z \, \mathrm{d}x = 2 \int_0^1 \int_0^{1-x} 1 - z \, \mathrm{d}z \, \mathrm{d}x = 2 \int_0^1 1 - x - \frac{(1-x)^2}{2} \, \mathrm{d}x = \int_0^1 1 - x^2 \, \mathrm{d}x = \frac{2}{3}$$

Problem 2 (20 points).

Use cylindrical coordinates to evaluate the following integral

a) $\iiint_E x^2 \, \mathrm{d}V$, where E s the solid that lies within the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$, and below the cone $z^2 = 4x^2 + 4y^2$.

$$\int_0^{2\pi} \int_0^1 \int_0^{2r} r^2 \cos^2 \varphi r \, \mathrm{d}z \, \mathrm{d}r \, \mathrm{d}\varphi = \int_0^{2\pi} \int_0^1 2r^4 \cos^2 \varphi \, \mathrm{d}r \, \mathrm{d}\varphi = \frac{1}{5} \int_0^{2\pi} 1 + \cos 2\varphi \, \mathrm{d}\varphi = \frac{2\pi}{5}$$

b) $\int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{\sqrt{x^2+y^2}}^{2} xz \, \mathrm{d}z \, \mathrm{d}x \, \mathrm{d}y$

$$\int_0^2 \int_0^{2\pi} \int_r^2 r \cos \varphi z r \, \mathrm{d}z \, \mathrm{d}\varphi \, \mathrm{d}r = \frac{1}{2} \int_0^2 \int_0^{2\pi} (4r^2 - r^4) \cos \varphi \, \mathrm{d}\varphi \, \mathrm{d}r = 0$$

Problem 3 (20 points).

Use spherical coordinates to evaluate the following integral

a) $\displaystyle\iiint_E xe^{x^2+y^2+z^2}\,\mathrm{d}V$, where E is the portion of the unit ball $x^2+y^2+z^2 \le 1$ and lies in the first octant.

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 r^3\cos\theta\sin^2\varphi e^{r^2}\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\cos\theta\sin^2\varphi\int_0^1 ue^u\,\mathrm{d}u\,\mathrm{d}\varphi\,\mathrm{d}\theta$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}}\cos\theta\int_0^{\frac{\pi}{2}} 1-\cos 2\varphi\,\mathrm{d}\varphi\,\mathrm{d}\theta = \frac{\pi}{8}\int_0^{\frac{\pi}{2}}\cos\theta\,\mathrm{d}\theta = \frac{\pi}{8}$$

b) $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2-\sqrt{4-x^2-y^2}}^{2+\sqrt{4-x^2-y^2}}(x^2+y^2+z^2)^{\frac{3}{2}}\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$

$$0 \le \theta \le 2\pi, 0 \le \varphi \le \frac{\pi}{2}, 0 \le r \le 4\cos\varphi$$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{4\cos\varphi} r^5\sin\varphi\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\theta = \frac{4^6}{6}\int_0^{2\pi}\int_0^{\frac{\pi}{2}}(\cos\varphi)^6\sin\varphi\,\mathrm{d}\varphi\,\mathrm{d}\theta = \frac{4^6}{6\cdot 7}\int_0^{2\pi}1\,\mathrm{d}\theta = \frac{4^6\pi}{21}$$

Problem 4 (20 points).

Find the volume

a) of the region E bounded by the paraboloids $z = x^2+y^2$ and $z = 36-3x^2-3y^2$.

$$x^2+y^2 = 36-3x^2-3y^2 \implies x^2+y^2 = 3^2$$

$$\int_0^{2\pi}\int_0^3\int_{r^2}^{36-3r^2} r\,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\varphi = \int_0^{2\pi}\int_0^3 36r-4r^3\,\mathrm{d}r\,\mathrm{d}\varphi = \int_0^{2\pi}81\,\mathrm{d}\varphi = 162\pi$$

b) of the solid that lies above the cone $\varphi = \pi/3$ and below the sphere $\rho = 4\cos(\varphi)$.

$$0 \le \theta \le 2\pi, 0 \le \varphi \le \frac{\pi}{3}, 0 \le r \le 4\cos\varphi$$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{4\cos\varphi} r^2\sin\varphi\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}\theta = \frac{4^3}{3}\int_0^{2\pi}\int_0^{\frac{\pi}{3}}\cos^3\varphi\sin\varphi\,\mathrm{d}\varphi\,\mathrm{d}\theta = 5\int_0^{2\pi}1\,\mathrm{d}\theta = 10\pi$$

Problem 5 (20 points).

By making an appropriate change of variables, evaluate

a) $\displaystyle\iint_R (x+y)e^{x^2-y^2}\,\mathrm{d}A$, where R is the rectangle enclosed by the lines $x-y=0$, $x-y=2$, $x+y=0$, and $x+y=3$.

$$u = x+y, v = x-y \implies x = \frac{1}{2}(u+v), y = \frac{1}{2}(u-v) \implies \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$$\int_0^3\int_0^2 \frac{1}{2}ue^{uv}\,\mathrm{d}v\,\mathrm{d}u = \frac{1}{2}\int_0^3 e^{2u}-1\,\mathrm{d}u = \frac{e^6-7}{4}$$

b) $\displaystyle\iint_D xy\,\mathrm{d}A$, where D is the region in the first quadrant bounded by the lines $y = x$, $y = 3x$ and the hyperbolas $xy = 1$, $xy = 3$.

$$u = xy, v = \frac{y}{x}, x, y \geq 0 \implies x = \sqrt{\frac{u}{v}}, y = \sqrt{uv} \implies \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2\sqrt{uv}} & -\frac{\sqrt{u}}{2v\sqrt{v}} \\ \frac{\sqrt{v}}{2\sqrt{u}} & \frac{1\sqrt{u}}{2\sqrt{v}} \end{vmatrix} = \frac{1}{2v}$$

$$\int_1^3 \int_1^3 \frac{u}{2v}\,\mathrm{d}u\,\mathrm{d}v = 2\int_1^3 \frac{1}{v}\,\mathrm{d}v = 2\ln 3$$