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CS102 Calculus 3 Section G - Homework 9

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Problem 1 (20 points).

Evaluate the line integral, where C is the given curve.

a) $\int_C y^3\,\mathrm{d}s$, $C: x = t^3, y = t, 0 \le t \le 2$

$$\int_0^2 t^3\sqrt{9t^4+1}\,\mathrm{d}t = \frac{1}{36}\int_0^2 \sqrt{9t^4+1}\,\mathrm{d}(9t^4+1) = \frac{1}{54}\left((9\cdot 2^4+1)^{\frac{3}{2}}-1\right) = \frac{145^{\frac{3}{2}}-1}{54}$$

b) $\int_C xyz\,\mathrm{d}s$, $C: x = 2\sin(t), y = t, z = -2\cos(t), 0 \le t \le \pi$

$$\int_0^\pi -4t\sin(t)\cos(t)\sqrt{5}\,\mathrm{d}t = \sqrt{5}\,[t\cos(2t)]_0^\pi - \sqrt{5}\int_0^\pi \cos(2t)\,\mathrm{d}t = \pi\sqrt{5}$$

c) $\int_C (x+2y)\,\mathrm{d}x + x^2\,\mathrm{d}y$, C consists of the line segments from $(0,0)$ to $(2,1)$ and from $(2,1)$ to $(3,0)$

$$x_1 = 2t, y_1 = t, x_2 = 2+t, y_2 = 1-t, t \in [0,1]$$

$$\int_0^1 8t+4t^2\,\mathrm{d}t + \int_0^1 4-t-(2+t)^2\,\mathrm{d}t = 3\int_0^1 t^2+t\,\mathrm{d}t = \frac{5}{2}$$

d) $\int_C y\,\mathrm{d}x + z\,\mathrm{d}y + x\,\mathrm{d}z$, $C: x = \sqrt{t}, y = t, z = t^2, 1 \le t \le 4$

$$\int_1^4 \frac{\sqrt{t}}{2}+t^2+2t^{\frac{3}{2}}\,\mathrm{d}t = \left(\frac{4^{\frac{3}{2}}}{3}+\frac{4^3}{3}+\frac{4\cdot 4^{\frac{5}{2}}}{5}\right) - \left(\frac{1^{\frac{3}{2}}}{3}+\frac{1^3}{3}+\frac{4\cdot 1^{\frac{5}{2}}}{5}\right) = \frac{722}{15}$$

Problem 2 (20 points).

Find the work done by the force field $\mathbf{F} = F(x,y)$ in moving an object along the trajectory C.

a) $F(x,y) = \{x, y+2\}$, $C: r(t) = (t-\sin(t), 1-\cos(t)), 0 \le t \le 2\pi$

$$\int_0^{2\pi} (t-\sin(t))(1-\cos(t)) + (3-\cos(t))\sin(t)\,\mathrm{d}t = \int_0^{2\pi} t - t\cos(t) + 2\sin(t) - 2\sin(t)\cos(t)\,\mathrm{d}t$$

$$\stackrel{*}{=} \left[\frac{t^2}{2} - t\sin(t) - 3\cos(t) + \frac{\cos(2t)}{2}\right]_0^{2\pi} = 2\pi^2$$

b) $F(x, y, z) = \{x - y^2, y - z^2, z - x^2\}$, C is the line segment from $(0, 0, 1)$ to $(2, 1, 0)$

$$x = 2t, y = t, z = 1 - t, t \in [0, 1]$$

$$\int_0^1 2(2t - t^2) + (t - (1 - t)^2) - (1 - t - (2t)^2)\,\mathrm{d}t = \int_0^1 8t + t^2 - 2\,\mathrm{d}t = 4 + \frac{1}{3} - 2 = \frac{7}{3}$$

Problem 3 (20 points).

Determine whether or no $\mathbf{F}$ is a conservative vector field. If it is, find a function f such that $F = \nabla f$.

a) $F(x, y) = \{e^x \cos(y), e^x \sin(y)\}$

$$-e^x \sin(y) \neq e^x \sin(y) \implies \mathbf{F} \text{ is not conservative}$$

b) $F(x, y) = \{ye^x + \sin(y), e^x + x\cos(y)\}$

$$e^x + \cos(y) = e^x + \cos(y) \text{ and } P'_x(x, y) = ye^x, Q'_y(x, y) = -x\sin(y) \text{ are continous}$$

$$\therefore \mathbf{F} \text{ is conservative} \implies \exists f, \text{ s.t. } \mathbf{F} = \nabla f$$

$$\begin{cases} P = f'_x \implies f = ye^x + x\sin(y) + c_1 \\ Q = f'_y \implies f = ye^x + x\sin(y) + c_2 \end{cases} \implies f = ye^x + x\sin(y) + c, c \in \mathbb{R}$$

Problem 4 (20 points).

For the given vector field $\mathbf{F}$ find a function f such that $\mathbf{F} = \nabla f$ and use it to evaluate $\displaystyle\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.

a) $F(x, y) = \{xy^2, x^2y\}$, $C : \mathbf{r}(t) = \left(t + \sin\left(\frac{\pi}{2}t\right), t + \cos\left(\frac{\pi}{2}t\right)\right), 0 \leq t \leq 1$

$$2xy = 2xy, P'_y, Q'_x \text{ are continuous} \implies \exists f, \mathbf{F} = \nabla f$$

$$\begin{cases} P = f'_x \implies f = \frac{x^2y^2}{2} + c_1 \\ Q = f'_y \implies f = \frac{x^2y^2}{2} + c_2 \end{cases} \implies f = \frac{x^2y^2}{2} + c, c \in \mathbb{R}$$

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \nabla f \cdot \mathrm{d}\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = 2$$

b) $F(x, y, z) = \{yz, xz, xy + 2z\}$, C is the line segment from $(1, 0, -2)$ to $(4, 6, 3)$

$$1 = 1 = 1, P'_x, Q'_y, R'_z \text{ are continuous} \implies \exists f, \mathbf{F} = \nabla f$$

$$\begin{cases} P = f'_x \implies f = xyz + c_1 \\ Q = f'_y \implies f = xyz + c_2 \\ R = f'_z \implies f = xyz + z^2 \end{cases} \implies f = xyz + z^2 + c, c \in \mathbb{R}$$

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_C \nabla f \cdot \mathrm{d}\mathbf{r} = f(\mathbf{r}(1)) - f(\mathbf{r}(0)) = 81$$

Problem 5 (20 points).

Use Green's theorem to evaluate $\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.

a) $F(x, y) = \{y + e^{\sqrt{x}}, 2x + \cos(y^2)\}$, C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$ oriented counterclockwise

$$P, Q \text{ are continuous} \implies \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^1 \int_{x^2}^{\sqrt{x}} 2 - 1 \,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \sqrt{x} - x^2 \,\mathrm{d}x = \frac{1}{3}$$

b) $F(x, y) = \{y\cos(x) - xy\sin(x), xy + x\cos(x)\}$, C is the triangle from $(0, 0)$ to $(0, 4)$ to $(2, 0)$ to $(0, 0)$

$$P, Q \text{ are continuous} \implies \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\int_0^4 \int_0^{4-2x} (\cos(x) - x\sin(x)) - (y + \cos(x) - x\sin(x)) \,\mathrm{d}y\,\mathrm{d}x$$

$$= -\int_0^2 \int_0^{4-2x} y \,\mathrm{d}y\,\mathrm{d}x = -2\int_0^2 (2 - x)^2 \,\mathrm{d}x = -2\left(8 - 8 + \frac{8}{3}\right) = -\frac{16}{3}$$

Problem 6 (10 points).

Determine whether or not the given set D is simply connected. If it is not, give an example of a vector field $\mathbf{F} = \{P(x, y), Q(x, y)\}$ such that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ holds at any point of D, and two curves C_1 and C_2 contained in D with the same initial and terminal points such that $\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.

a) $D = \{(x, y) | 1 \le x^2 + y^2 \le 4, y \ge 0\}$

$$D \text{ has no holes} \implies D \text{ is simply connected}$$

b) $D = \{(x, y) | (x - 2)^2 + (y - 3)^2 > 1\}$

$$D \text{ is not simply connected, as it has a hole } (x - 2)^2 + (y - 3)^2 \le 1$$

$$\mathbf{F} := \{-\frac{y - 3}{(x - 2)^2 + (y - 3)^2}, \frac{x - 2}{(x - 2)^2 + (y - 3)^2}\}, \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\begin{cases} C_1 : \mathbf{r}(t) = (2\cos(t) + 2, 2\sin(t) + 3) \\ C_2 : \mathbf{r}(t) = (2\cos(t) + 2, -2\sin(t) + 3) \end{cases}, t \in [0, \pi]$$

$$\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^\pi \sin^2 t + \cos^2 t \,\mathrm{d}t = \pi \neq \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^\pi -\sin^2 t - \cos^2 t \,\mathrm{d}t = -\pi$$

Used things

Denote P and Q functions of x and y, s.t. $\mathbf{F} = \{P(x, y), Q(x, y)\}$

$$* \int t\cos(t)\,\mathrm{d}t = \int t\,\mathrm{d}\sin(t) \overset{IbP}{=} t\sin(t) - \int \sin(t)\,\mathrm{d}t = t\sin(t) + \cos(t) + c$$