Digital version

# CS104 Linear Algebra Section D - Homework 10

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## Exercise 1 (15 points).

Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ with the standard matrix $[T] = \begin{bmatrix} 0 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix}$, the linear transformation $S : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $S\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x-2y \\ x \end{bmatrix}$ and $L : \mathbb{R}^3 \to \mathbb{R}^2$, defined by $L\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+z \\ 2z \end{bmatrix}$

a) Use the matrix method to compute $(S \circ L)\begin{bmatrix} x \\ y \\ z \end{bmatrix}$;

Using the standard basis for $S$ and $L$, we get:

$$[S] = \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix} \text{ and } [L] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \implies [S \circ L] = [S][L] = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 0 & 1 \end{bmatrix}$$

$$(S \circ L)\begin{bmatrix} x \\ y \\ z \end{bmatrix} = [S \circ L]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-3z \\ x+z \end{bmatrix}$$

b) Show that $S$ is invertible and find the standard matrix of $S^{-1}$;

$$\det([S]) = 2 \neq 0 \implies \exists[S^{-1}] \implies \exists S^{-1}, \text{ and } [S^{-1}] = \frac{1}{\det([S])}\begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

c) Find the standard matrix of the linear transformation $F = S^{-1} \circ T$

$$[F] = [S^{-1}][T] = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -\frac{1}{2} & \frac{3}{2} & 04 \end{bmatrix}$$

## Exercise 2 (15 points).

Consider the linear transformation $T : P_3(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ defined by $T(a+bx+cx^2+dx^3) = \begin{bmatrix} 3b+c & 0 \\ 0 & 2d-c \end{bmatrix}$.

a) Describe range($T$);

$$\text{range}(T) = \left\{ \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \text{ s.t. } \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 3b+c & 0 \\ 0 & 2d-c \end{bmatrix}, \forall a,b,c,d \in \mathbb{R} \right\}$$

$$= \left\{ \begin{bmatrix} 3b & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} c & 0 \\ 0 & -c \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2d \end{bmatrix}, \forall a,b,c,d \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\because \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ are basis for range}(T)$$

b) Find rank($T$) and use the rank theorem to compute nullity($T$);

$$\text{rank}(T) = \dim \text{range}(T) = 2$$

$$\text{rank}(T) + \text{nullity}(T) = \dim P_3(\mathbb{R}) \implies \text{nullity}(T) = 2$$

c) Describe Ker($T$).

$$\text{Ker}(T) = \left\{ a + bx + cx^2 + dx^3, \text{ s.t. } \begin{bmatrix} 3b+c & 0 \\ 0 & 2d-c \end{bmatrix} = 0, a,b,c,d, \in \mathbb{R} \right\}$$

$$= \left\{ a + bx + cx^2 + dx^3, \text{ s.t. } \begin{cases} 0a + 3b + c + 0d = 0 \\ 0a + 0b - c + 2d = 0 \end{cases}, a,b,c,d \in \mathbb{R} \right\}$$

$$\left[\begin{array}{cccc|c} 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right] \implies \begin{cases} b = -\frac{1}{3}d \\ c = 2d \end{cases}$$

$$\text{Ker}(T) = \left\{ a - \frac{1}{3}ax + 2dx^2 + dx^3, \text{ s.t. } , a,d \in \mathbb{R} \right\} = \text{span}\left\{ 1 - \frac{1}{3}x, 2x^2 + x^3 \right\}$$

The two vectors are linearly independent, which implies that they form basis for Ker($T$).

## Exercise 3 (15 points).

For the given linear transformation $S : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ defined by $S(A) = A - A^T$, where $A$ is a fixed $2 \times 2$ matrix, find Ker($S$), basis for Ker($S$), Rng($S$), basis for Rng($S$) and verify the **rank-nullity theorem**.

$$A := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies S(A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix}, \forall a,b,c,d \in \mathbb{R}$$

$$\text{Ker}(S) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ s.t. } \begin{bmatrix} 0 & b-c \\ -(b-c) & 0 \end{bmatrix} = 0, a,b,c,d \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

The matrices are linearly independent, hence they form basis for the Ker($S$).

$$\text{Rng}(S) = \left\{ \begin{bmatrix} 0 & b-c \\ c-b & 0 \end{bmatrix}, \forall b,c \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 0 & b-c \\ -(b-c) & 0 \end{bmatrix}, \forall b,c \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

$$\text{rank}(S) + \text{nullity}(S) = \dim M_{2\times 2}(\mathbb{R}) \implies \dim \text{Rng}(S) + \dim \text{Ker}(S) = \dim M_{2\times 2}(\mathbb{R}) \implies 1 + 3 = 4$$

## Exercise 4 (15 points).

Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^3$ and $T_2 : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformations with matricies $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 & 3 \\ 3 & 0 & 1 \end{bmatrix}$ respectively. Find $T_2 \circ T_1$, $\mathrm{Ker}(T_2 \circ T_1)$ and $\mathrm{Rng}(T_2 \circ T_1)$.

$$[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 0 & -1 & 3 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 6 & -4 \end{bmatrix}$$

$$T_2 \circ T_1 : \mathbb{R}^2 \to \mathbb{R}^2, \text{ s.t. } (T_2 \circ T_1) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -a - 3b \\ 6a - 4b \end{bmatrix}$$

$$\mathrm{Ker}(T_2 \circ T_1) = \left\{ \begin{bmatrix} a \\ b \end{bmatrix}, \text{ s.t. } (T_2 \circ T_1) \begin{bmatrix} a \\ b \end{bmatrix} = \mathbf{0}, a, b \in \mathbb{R} \right\} = \{\mathbf{0}\}$$

$$\mathrm{Rng}(T_2 \circ T_1) = \left\{ \begin{bmatrix} -a - 3b \\ 6a - 4b \end{bmatrix}, \forall a, b \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} -a \\ 6a \end{bmatrix} + \begin{bmatrix} -3b \\ -4b \end{bmatrix}, \forall a, b \in \mathbb{R} \right\} = \mathrm{span} \left\{ \begin{bmatrix} -1 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$$

The two vectors are linearly independent, which implies that they form basis for $\mathrm{Rng}(T_2 \circ T_1)$.

## Exercise 5 (15 points).

Let $T_1 : M_2(\mathbb{R}) \to P_2(\mathbb{R})$, $T_2 : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ be two linear transformations defined by $T_1\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + (b - 2c)x + (d + a)x^2$ and $T_2(p(x)) = \int_0^x p(t)\ \mathrm{d}t$. Determine $T_2 \circ T_1$, $\mathrm{Ker}(T_2 \circ T_1)$, $\mathrm{Rng}(T_2 \circ T_1)$, $\mathrm{nullity}(T_2 \circ T_1)$ and $\mathrm{rank}(T_2 \circ T_1)$.

$$T_2(a + (b - 2c)x + (d + a)x^2) = \int_0^x (a + (b - 2c)t + (d + a)t^2)\ \mathrm{d}t = \left[ at + (b - 2c)\frac{t^2}{2} + (d + a)\frac{t^3}{3} \right]_0^x$$

$$\therefore T_2 \circ T_1 : M_2(\mathbb{R}) \to P_3(\mathbb{R}, \text{ s.t. } (T_2 \circ T_1) \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ax + \frac{(b - 2c)}{2}x^2 + \frac{(d + a)}{3}x^3, \forall a, b, c, d \in \mathbb{R}$$

$$\mathrm{Ker}(T_2 \circ T_1) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ s.t. } ax + \frac{(b - 2c)}{2}x^2 + \frac{(d + a)}{3}x^3 = 0, a, b, c, d \in \mathbb{R} \right\}$$

$$\begin{cases} a = 0 \\ \frac{b-2c}{2} = 0 \\ \frac{d+a}{3} = 0 \end{cases} \implies \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array} \right] \longrightarrow \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

$$\mathrm{Ker}(T_2 \circ T_1) = \mathrm{span} \left\{ \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \right\}$$

$$\mathrm{Rng}(T_2 \circ T_1) = \left\{ ax + \frac{(b - 2c)}{2}x^2 + \frac{(d + a)}{3}x^3, \forall a, b, c, d \in \mathbb{R} \right\} = \mathrm{span} \left\{ x + \frac{x^3}{3}, \frac{1}{2}x^2, \frac{-2}{2}x^2, \frac{1}{3}x^3 \right\}$$

$$\mathrm{Rng}(T_2 \circ T_1) = \mathrm{span} \left\{ 3x + x^3, x^2, x^3 \right\} \text{ and they form basis}$$

$$\mathrm{nullity}(T) = \dim \mathrm{Ker}(T) = 1$$

$$\mathrm{rank}(T) = \dim \mathrm{Rng}(T) = 3$$

## Exercise 6 (15 points).

For the given linear transformation find $\text{Ker}(T)$ and $\text{Rng}(T)$, and hence, determine whether the given transformation is one-to-one, onto, both, or neither. Find a formula for the inverse linear transformation $T^{-1}$ or explain why it does not exist.

a) Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$;

$$\mathbf{x} := \begin{bmatrix} a \\ b \end{bmatrix} \implies A\mathbf{x} = \begin{bmatrix} 4a+2b \\ a+3b \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} \frac{3}{10} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{2}{5} \end{bmatrix}$$

$$\text{Rng}(T) = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}, \text{Ker}(T) = \{\mathbf{0}\}$$

$$\therefore T \text{ is one-to-one and onto } \implies \exists T^{-1}, T^{-1} \begin{bmatrix} a \\ b \end{bmatrix} = A^{-1} \begin{bmatrix} a \\ b \end{bmatrix}$$

b) Define $T : P_1(\mathbb{R}) \to P_1(\mathbb{R})$ by $T(ax+b) = (b-2a)x + (2b+a)$;

$$\text{Rng}(T) = \text{span}\left\{1-2x, 2+x\right\}, \text{Ker}(T) = \{\mathbf{0}\}$$

$$\therefore T \text{ is one-to-one and onto } \implies \exists T^{-1}, T^{-1}(ax+b) = \frac{2a-b}{3}x + \frac{7a-2b}{3}$$

c) Define $T : P_2(\mathbb{R}) \to P_1(\mathbb{R})$ by $T(ax^2+bx+c) = (a+2b)x + c$.

$$\text{Rng}(T) = \text{span}\left\{1, x\right\}, \text{Ker}(T) = \text{span}\left\{-2x^2+x\right\}$$

$$\therefore T \text{ is onto but is not one-to-one } \implies \nexists T^{-1}$$

## Exercise 7 (10 points).

Determine an isomorphism between $\mathbb{R}^3$ and the subspace of $M_{2\times2}(\mathbb{R})$ consisting of all upper triangular matrices.

$$T : \mathbb{R}^3 \to M_{2\times2}(\mathbb{R}), \text{ s.t. } T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$$