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CS104 Linear Algebra Section D - Homework 11

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Exercise 1 (30 points).

Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, given by its standard matrix $A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$.

a) Show that the vectors $\mathbf{v_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ are eigenvectors for A, and find the corresponding eigenvalues.

$$A\mathbf{x} = \lambda\mathbf{x} \implies A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} \implies \lambda_1 = 3$$

$$A\mathbf{x} = \lambda\mathbf{x} \implies A\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} = -2\begin{bmatrix} -1 \\ 2 \end{bmatrix} \implies \lambda_2 = -2$$

b) Compute $T^{100}\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \to \left[\begin{array}{cc|c} 2 & -1 & -1 \\ 1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -5 & -1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -\frac{2}{5} \\ 0 & 1 & \frac{1}{5} \end{array}\right]$$

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix} = -\frac{2}{5}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$T^{100}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = [T]^{100}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{100}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{100}\left(-\frac{2}{5}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$$

$$= -\frac{2}{5}\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{100}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}^{100}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = -\frac{2}{5}(3)^{100}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{5}(-2)^{100}\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\therefore T^{100}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = -\frac{1}{5}\left(2 \cdot 3^{100}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2^{100}\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right)$$

Exercise 2 (40 points).

Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$. Compute

a) the characteristic polynomial of A,

$$p(\lambda) = \det(A - \lambda I) = \det\left(\begin{bmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & -2 \\ 1 & 0 & 1-\lambda \end{bmatrix}\right)$$

$$= (3-\lambda)\begin{vmatrix} 1-\lambda & -2 \\ 0 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda)(1-\lambda) \implies p(\lambda) = -\lambda^3 + 5\lambda^2 - 7\lambda + 3)$$

b) the eigenvalues of A,

$$p(\lambda) = -\lambda^3 + 5\lambda^2 - 7\lambda + 3 = (3-\lambda)(1-\lambda)(1-\lambda) = 0$$

$$\lambda_1 = 3, \lambda_2 = 1$$

c) a basis for each eigenspace of A,

$$\begin{bmatrix} 3-3 & 0 & 0 \\ 0 & 1-3 & -2 \\ 1 & 0 & 1-3 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & -2 \\ 1 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \implies \begin{cases} x_1 = 2x_3 \\ x_2 = -x_3 \end{cases}$$

$$E_{\lambda_2} = \text{null}\begin{bmatrix} 3-3 & 0 & 0 \\ 0 & 1-3 & -2 \\ 1 & 0 & 1-3 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}\right\}$$

$$\begin{bmatrix} 3-1 & 0 & 0 \\ 0 & 1-1 & -2 \\ 1 & 0 & 1-1 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -2 \\ 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \implies \begin{cases} x_1 = 0 \\ x_3 = 0 \end{cases}$$

$$E_{\lambda_2} = \text{null}\begin{bmatrix} 3-1 & 0 & 0 \\ 0 & 1-1 & -2 \\ 1 & 0 & 1-1 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}$$

d) the algebraic and geometric multiplicity of each eigenvalue of A.

$$\lambda_1 \to 1, \dim(E_{\lambda_1}) = 1$$

$$\lambda_2 \to 2, \dim(E_{\lambda_2}) = 1$$

Exercise 3 (30 points).

a) Find the eigenvalues of $A = \begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}$ over the complex numbers $\mathbb{C}$ and determine all eigenvectors corresponding to these eigenvalues.

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & -3 \\ 1 & -\lambda \end{bmatrix} = (2-\lambda)(-\lambda) + 3 = \lambda^2 - 2\lambda + 3 = 0$$

$$\lambda_1 = \frac{2 + \sqrt{4 - 4 \cdot 3}}{2} = 1 + i\sqrt{2}, \lambda_2 = \frac{2 - \sqrt{4 - 4 \cdot 3}}{2} = 1 - i\sqrt{2}$$

$$A - \lambda_1 I = \begin{bmatrix} 2 - 1 - i\sqrt{2} & -3 \\ 1 & -1 - i\sqrt{2} \end{bmatrix} \to \begin{bmatrix} 1 & -1 - i\sqrt{2} \\ 1 - i\sqrt{2} & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 - i\sqrt{2} \\ 0 & 0 \end{bmatrix}$$

$$\therefore E_{\lambda_1} = \text{span} \left\{ \begin{bmatrix} 1 + i\sqrt{2} \\ 1 \end{bmatrix} \right\}$$

$$A - \lambda_2 I = \begin{bmatrix} 2 - 1 + i\sqrt{2} & -3 \\ 1 & -1 + i\sqrt{2} \end{bmatrix} \to \begin{bmatrix} 1 & -1 + i\sqrt{2} \\ 1 + i\sqrt{2} & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 + i\sqrt{2} \\ 0 & 0 \end{bmatrix}$$

$$\therefore E_{\lambda_1} = \text{span} \left\{ \begin{bmatrix} 1 - i\sqrt{2} \\ 1 \end{bmatrix} \right\}$$

b) Find the eigenvalues of $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ over the $\mathbb{Z}_3$ and determine all eigenvectors corresponding to these eigenvalues.

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = (2-\lambda)(2-\lambda) - 1 = \lambda^2 - \lambda = 0 \implies \lambda = 0$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \implies E_{\lambda_1} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$