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CS104 Linear Algebra Section D - Homework 1

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Exercise 1 (20 points).

Consider the vectors

$$\vec{u} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ -2 \end{bmatrix}; \vec{v} = \begin{bmatrix} 2 \\ 1 \\ -2 \\ 0 \end{bmatrix}; \vec{w} = \begin{bmatrix} 2 \\ -3 \\ 1/2 \\ 1 \end{bmatrix}; \vec{p} = \begin{bmatrix} -4 \\ -2 \\ 4 \\ 0 \end{bmatrix}$$

(a) Check if $\vec{u}$ an $\vec{w}$ are parallel. Check if $\vec{v}$ an $\vec{p}$ are parallel.

Two vectors are parallel if one of them is equal to the other multiplied by a constant. Assume $\vec{u}$ and $\vec{w}$ are parallel.

$$\therefore \vec{u} = c \cdot \vec{w}, \text{where c} \in \mathbb{R} \implies 1 = 2 \cdot c, 3 = -3 \cdot c \implies c = 1/2 \text{ and } c = -1$$
$$\text{We have a contradiction } \implies \vec{u} \text{ and } \vec{v} \text{ are not parallel.}$$

Assume $\vec{v}$ and $\vec{p}$ are parallel.

$$\therefore \vec{v} = c \cdot \vec{p}, \text{where c} \in \mathbb{R} \implies 2 = -4 \cdot c, 1 = -2 \cdot c, -2 = 4 \cdot c, 0 = 0 \cdot c \implies c = -\frac{1}{2}$$
$$\vec{v} \text{ and } \vec{p} \text{ are parallel.}$$

(b) Compute the vector $\vec{u} + 2\vec{v} - 3\vec{p}$. Write the answer in horizontal (row-vector) and vertical (column-vector) forms.

$$\vec{u} + 2\vec{v} - 3\vec{p} = \begin{bmatrix} 1 + 2 \cdot 2 - 3 \cdot (-4) \\ 3 + 2 \cdot 1 - 3 \cdot (-2) \\ 0 + 2 \cdot (-2) - 3 \cdot 4 \\ -2 + 2 \cdot 0 - 3 \cdot 0 \end{bmatrix} = \begin{bmatrix} 17 \\ 11 \\ -16 \\ -2 \end{bmatrix} = \begin{bmatrix} 17, 11, -16, -2 \end{bmatrix}$$

(c) Compute the dot products $\vec{u} \cdot \vec{v}, \vec{u} \cdot \vec{w}, \vec{v} \cdot \vec{w}$.

$$\vec{u} \cdot \vec{v} = 1 \cdot 2 + 3 \cdot 1 + 0 \cdot (-2) + (-2) \cdot 0 = 5$$
$$\vec{u} \cdot \vec{w} = 1 \cdot 2 + 3 \cdot (-3) + 0 \cdot 1/2 + (-2) \cdot 1 = -9$$
$$\vec{v} \cdot \vec{w} = 2 \cdot 2 + 1 \cdot (-3) + (-2) \cdot 1/2 + 0 \cdot 1 = 0$$

(d) Using the previous point determine whether the angles between $\vec{u}$ and $\vec{v}$, $\vec{u}$ and $\vec{w}$, $\vec{v}$ and $\vec{w}$ are acute, obtuse, or right angles?
$\because \vec{u} \cdot \vec{v} > 0 \implies$ the angle between $\vec{u}$ and $\vec{v}$ is acute
$\because \vec{u} \cdot \vec{w} < 0 \implies$ the angle between $\vec{u}$ and $\vec{w}$ is obtuse
$\because \vec{v} \cdot \vec{w} = 0 \implies$ the angle between $\vec{v}$ and $\vec{w}$ is a right angle

(e) Using the previous points find all orthogonal pairs of vectors if there are any.

$\vec{v}$ and $\vec{p}$ are parallel and $\vec{v}$ and $\vec{w}$ are orthogonal. $\implies$ $\vec{p}$ and $\vec{w}$ are orthogonal.

(f) Compute the lengths/norms of $\vec{u}$ and $\vec{v}$.

$$\|\vec{u}\| = \sqrt{(1)^2 + (3)^2 + (0)^2 + (-2)^2} = \sqrt{14}$$

$$\|\vec{v}\| = \sqrt{(2)^2 + (1)^2 + (-2)^2 + (0)^2} = 3$$

(g) Find the angle between $\vec{u}$ and $\vec{v}$.

$$\theta = arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right) = arccos\left(\frac{5}{\sqrt{14} \cdot 3}\right) \approx 63.5^\circ$$

(h) Normalize $\vec{u}$ and $\vec{v}$. What is the angle between normalized vectors?

$$\vec{u}_1 = \frac{1}{\|\vec{u}\|}\vec{u} = \begin{bmatrix} 1/\sqrt{14} \\ 3/\sqrt{14} \\ 0 \\ -2/\sqrt{14} \end{bmatrix}$$

$$\vec{v}_1 = \frac{1}{\|\vec{v}\|}\vec{v} = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \\ 0 \end{bmatrix}$$

$$\theta \approx 63.5^\circ$$

(i) Use the points above and the properties of length to compute the length of the vector $-3\vec{u}$

$$\|-3\vec{u}\| = 3 \cdot \|\vec{u}\| = 3\sqrt{14}$$

(j) Find the distance $d(\vec{u}; \vec{v})$

$$d(\vec{u}; \vec{v}) = \|\vec{u} - \vec{v}\| = \left\| \begin{bmatrix} 1 \\ 3 \\ 0 \\ -2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ -2 \\ 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -1 \\ 2 \\ 2 \\ -2 \end{bmatrix} \right\| = \sqrt{13}$$

Exercise 2 (20 points).

Let $A = (4, -1, 3)$, $B = (6, -2, 4)$, $C = (2, 0, 5)$ be points in $\mathbb{R}^3$

(a) Compute the projection $\vec{AB}$ on $\vec{AC}$

$$\vec{AB} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \vec{AC} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

$$\vec{AD} := proj_{\vec{AC}}(\vec{AB}) = \frac{\vec{AB} \cdot \vec{AC}}{\vec{AC} \cdot \vec{AC}} \vec{AC} = \frac{(-2) \cdot 2 + 1 \cdot (-1) + 2 \cdot 1}{(-2) \cdot (-2) + 1 \cdot 1 + 2 \cdot 2} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix}$$

(b) Find the area of the triangle ABC

$$S_{ABC} = \frac{\left\|\vec{AC}\right\| \cdot \left\|\vec{BD}\right\|}{2} = \frac{\left\|\vec{AC}\right\| \cdot \sqrt{\left\|\vec{AB}\right\|^2 - \left\|\vec{AD}\right\|^2}}{2} = \frac{\left\|\vec{AC}\right\| \cdot \sqrt{\left\|\vec{AB}\right\|^2 - \left\|\vec{AD}\right\|^2}}{2}$$

$$= \frac{3 \cdot \sqrt{6 - 1}}{2} = \frac{3\sqrt{15}}{2}$$

Exercise 3 (30 points).

(a) Prove that $\vec{u} \cdot \vec{v} = \frac{1}{4}\|\vec{u} + \vec{v}\|^2 - \frac{1}{4}\|\vec{u} - \vec{v}\|^2$

$$\frac{1}{4}\|\vec{u} + \vec{v}\|^2 - \frac{1}{4}\|\vec{u} - \vec{v}\|^2 = \frac{1}{4}(\vec{u} + \vec{v})^2 - \frac{1}{4}(\vec{u} - \vec{v})^2$$

$$= \frac{1}{4}(\vec{u} \cdot \vec{u} + 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v} - \vec{u} \cdot \vec{u} + 2\vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{v})$$

$$= \frac{1}{4}(4\vec{u} \cdot \vec{v}) = \vec{u} \cdot \vec{v}$$

(b) If $\|\vec{u}\| = 2, \|\vec{v}\| = \sqrt{3}$ and $\vec{u} \cdot \vec{v} = 1$, find $\|\vec{u} + \vec{v}\|$

$$\|\vec{u} + \vec{v}\| = \sqrt{(\vec{u} + \vec{v}) \cdot (\vec{u} + \vec{v})}$$

$$= \sqrt{\vec{u} \cdot \vec{u} + 2\vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{v}} = \sqrt{\|\vec{u}\|^2 + 2\vec{u} \cdot \vec{v} + \|\vec{v}\|^2}$$

$$= \sqrt{2^2 + 2 \cdot 1 + \sqrt{3}^2} = 3$$

(c) Prove that $\vec{u}$ is orthogonal to $\vec{v} - proj_{\vec{u}}(\vec{v})$ for all vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$, where $\vec{u} \neq 0$

$$proj_{\vec{u}}(\vec{v}) = \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \vec{u}, \vec{u} \text{ and } \vec{v} \text{ in } \mathbb{R}^n, \vec{u} \neq 0$$

$$\vec{u} \cdot (\vec{v} - proj_{\vec{u}}(\vec{v})) = \vec{u} \cdot \vec{v} - \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \vec{u} \cdot \vec{u} = \vec{u} \cdot \vec{v} - \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \|\vec{u}\|^2$$

$$= \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{v} = 0 \implies \vec{u} \text{ is orthogonal to } \vec{v} - proj_{\vec{u}}(\vec{v})$$

Exercise 4 (10 points).

Are there any vectors $\vec{u}$ and $\vec{v}$ with the conditions below? If not, explain why?

(a) $\|\vec{u}\| = 3, \|\vec{v}\| = 2, \vec{u} \cdot \vec{v} = -\sqrt{37}$

$$\theta = arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right) = arccos\left(\frac{-\sqrt{37}}{3 \cdot 2}\right)$$

$$\because \frac{-\sqrt{37}}{3 \cdot 2} < -1 \implies \nexists\theta \implies \text{such vectors cannot exist.}$$

(b) $\|\vec{u}\| = 2, \|\vec{v}\| = \sqrt(3), \vec{u} \cdot \vec{v} = 1$

$$\theta = arccos\left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right) = arccos\left(\frac{1}{2 \cdot \sqrt{3}}\right)$$

$$\because -1 < \frac{1}{2 \cdot \sqrt{3}} < 1 \implies \exists\theta \implies \text{such vectors do exist.}$$

Exercise 5 (20 points).

Write the equation of the line **l** passing through P = (2,3) with normal vector $\vec{n}$ = [1, -2] in

(a) General form
By the normal definition of a line in $\mathbb{R}^2$:

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}, \text{ where } \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}, (x, y) \in \mathbf{l} \text{ and } \vec{p} = \vec{OP}$$

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$1 \cdot x + (-2) \cdot y = 1 \cdot 2 + (-2) \cdot 3$$

$$x - 2y = -4$$

(b) Parametric form

$$\vec{x} = \vec{p} + t \cdot \vec{d}, t \in \mathbb{R}$$

$$\text{Take direction vector } \vec{d} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \because \vec{n} \cdot \vec{d} = 0$$

$$\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + t \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

$$\begin{cases} x = 2 + 2t \\ y = 3 + t \end{cases}, t \in \mathbb{R}$$