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CS104 Linear Algebra Section D - Homework 2

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Exercise 1 (40 points).

The plane $\mathcal{P}$ contains the points $A = (2, -1, 0)$ and $B = (1, 0, -3)$ and is perpendicular to the plane $\mathbb{Q}$ given by its general equation $x + 3y - 2z = 2$.

a) Find two nonparallel direction vectors for $\mathcal{P}$.

$$\because A, B \in \mathcal{P} \implies \vec{AB} \in \mathcal{P} \implies \mathbf{u} := \vec{AB} = \begin{bmatrix} 1-2 \\ 0-(-1) \\ -3-0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}.$$

$$\text{A normal vector of } \mathbb{Q} \text{ is } \mathbf{n} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \because \mathcal{P} \perp \mathbb{Q} \implies \mathbf{n} \text{ is a direction vector of } \mathcal{P}, \mathbf{v} := \mathbf{n}$$

$$\mathbf{u} = \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \text{ are direction vectors of } \mathcal{P}$$

b) Find the vector and the parametric equations of $\mathcal{P}$.

$$\mathbf{x} = \mathbf{p} + t\mathbf{u} + s\mathbf{v} \implies \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix} + s \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, x, y, z \in \mathcal{P}, s, t \in \mathbb{R}$$

$$\begin{cases} x = 2 + (-1)t + 1s \\ y = -1 + 1t + 3s \\ z = 0 + (-3)t + (-2)s \end{cases}, s, t \in \mathbb{R} \implies \begin{cases} x = 2 - t + s \\ y = -1 + t + 3s \\ z = -3t - 2s \end{cases}, s, t \in \mathbb{R}$$

c) Find a normal vector for $\mathcal{P}$.

$$\text{Take normal vector } \mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{bmatrix} 1 \cdot (-2) - (-3) \cdot 3 \\ (-3) \cdot 1 - (-1) \cdot (-2) \\ (-1) \cdot 3 - 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ -4 \end{bmatrix}$$

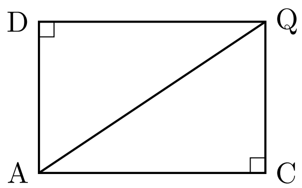
d) Find the normal and the general equations of $\mathcal{P}$.

$$\begin{bmatrix}7\\-5\\-4\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}7\\-5\\-4\end{bmatrix}\cdot\begin{bmatrix}2\\-1\\0\end{bmatrix}=7\cdot 2+(-5)\cdot(-1)+(-4)\cdot 0=19$$

$$7x-5y-4z=19$$

e) Find the distance $d(Q,\mathcal{P})$ from the point $Q=(3,-2,2)$ to $\mathcal{P}$.

Let C be a point on $\mathcal{P}$, such that $QC\perp\mathcal{P}$. Distance between a point and a plane is the perpendicular line, therefore QC will be the distance between Q and $\mathcal{P}$. Construct the diagram.



QC is perpendicular to $AC\in\mathcal{P}$. Take D, such that $\vec{AD}\parallel\mathbf{n}$ and $AD\perp DQ$. This implies that the triangles ACQ and ADQ are equal, which implies that $QC=AD$. Therefore $d(Q,\mathcal{P})=QC=AD=\left\|proj_{\mathbf{n}}\vec{AQ}\right\|$

$$\vec{AQ}=\begin{bmatrix}3-2\\(-2)-(-1)\\2-0\end{bmatrix}=\begin{bmatrix}1\\-1\\2\end{bmatrix},\mathbf{n}=\begin{bmatrix}7\\-5\\-4\end{bmatrix}$$

$$proj_{\mathbf{n}}\vec{AQ}=\frac{\vec{AQ}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\mathbf{n}=\frac{1\cdot 7+(-1)\cdot(-5)+2\cdot(-4)}{7\cdot 7+(-5)\cdot(-5)+(-4)\cdot(-4)}\begin{bmatrix}7\\-5\\-4\end{bmatrix}=\frac{2}{45}\begin{bmatrix}7\\-5\\-4\end{bmatrix}$$

$$\left\|proj_{\mathbf{n}}\vec{AQ}\right\|=\frac{2}{45}\sqrt{7^2+(-5)^2+(-4)^2}=\frac{2}{15}\sqrt{10}$$

Exercise 2 (40 points).

The line l passes through the point $P=(2,0,-3)$ and is perpendicular to the plane $\mathcal{P}$ given by its vector equation $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-2\\3\\1\end{bmatrix}+t\begin{bmatrix}-1\\3\\2\end{bmatrix}+s\begin{bmatrix}1\\-2\\5\end{bmatrix}\quad s,t\in\mathbb{R}$

a) Find two non-parallel normal vectors for l.

$$\because l\perp\mathcal{P}\implies l\text{ is perpendicular to direction vectors of }\mathcal{C}$$

$$l\perp\begin{bmatrix}-1\\3\\2\end{bmatrix}\text{ and }l\perp\begin{bmatrix}1\\-2\\5\end{bmatrix}$$

b) Find the normal and general equations of l.

$$\begin{cases} \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} \\ \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} \end{cases} \implies \begin{cases} \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -8 \\ \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -13 \end{cases} \implies \begin{cases} -x + 3y + 2z = -8 \\ x - 2y + 5z = -13 \end{cases}$$

c) Find a direction vector for l.

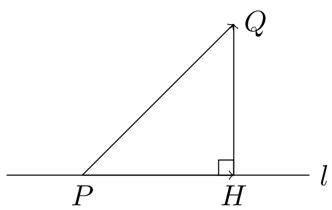
$$\because l \perp \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} \text{ and } l \perp \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} \implies l \parallel \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} \times \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} =: \mathbf{d}$$

$$\mathbf{d} = \begin{bmatrix} 3 \cdot 5 - 2 \cdot (-2) \\ 2 \cdot 1 - (-1) \cdot 5 \\ (-1) \cdot (-2) - 3 \cdot 1 \end{bmatrix} = \begin{bmatrix} 19 \\ 7 \\ -1 \end{bmatrix}$$

d) Find the vector and parametric equations of l.

$$\mathbf{x} = \mathbf{p} + r\mathbf{d}, r \in \mathbb{R} \implies \mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} + r \begin{bmatrix} 19 \\ 7 \\ -1 \end{bmatrix}, r \in \mathbb{R} \implies \begin{cases} x = 2 + 19r \\ y = 7r \\ z = -3 - r \end{cases}, r \in \mathbb{R}$$

e) Find the distance $d(Q, l)$ from the point $Q = (2, 1, -3)$ to l.



Take the projection PH of PQ on $l \implies HQ \perp PH$.We can find distance between Q and l by finding the distance of $\vec{HQ} = \vec{PQ} - \vec{PH} = \vec{PQ} - \text{proj}_l \vec{PQ}$.

$$\vec{PQ} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{proj}_l \vec{PQ} = \frac{\mathbf{d} \cdot \vec{PQ}}{\mathbf{d} \cdot \mathbf{d}} \mathbf{d} = \frac{19 \cdot 0 + 7 \cdot 1 + (-1) \cdot 0}{19 \cdot 19 + 7 \cdot 7 + (-1) \cdot (-1)} \begin{bmatrix} 19 \\ 7 \\ -1 \end{bmatrix} = \frac{7}{411} \begin{bmatrix} 19 \\ 7 \\ -1 \end{bmatrix}$$

$$d(Q, l) = \left\| \vec{PQ} - \text{proj}_l \vec{PQ} \right\| = \left\| \begin{bmatrix} 133/411 \\ 362/411 \\ -7/411 \end{bmatrix} \right\| = \frac{1}{411} \sqrt{133^2 + 362^2 + (-7)^2} = \frac{1}{411} \sqrt{148782} \approx 0.9385$$

Exercise 3 (20 points).

The line l passes through the point $P = (-3, 2, 3)$ and has direction vector $\mathbf{d} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$. For each of the following planes $\mathcal{P}$, determine whether l and $\mathcal{P}$ are parallel, perpendicular, or neither:

a) -3x + 9y - 6z = 11.

$$\mathbf{n} = \begin{bmatrix} -3 \\ 9 \\ -6 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = -3\mathbf{d} \implies \mathbf{n} \parallel \mathbf{d} \implies l \perp \mathcal{P}$$

b) 4x + 2y + z = 4.

$$\mathbf{n} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \implies \mathbf{n} \cdot \mathbf{d} = 4 \cdot 1 + 2 \cdot (-3) + 1 \cdot 2 = 0 \implies \mathbf{n} \perp \mathbf{d} \implies l \parallel \mathcal{P}$$

c) 2x + 5y - 3z = 6.

$$\mathbf{n} = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix} \implies \mathbf{n} \cdot \mathbf{d} = 2 \cdot 1 + 5 \cdot (-3) + (-3) \cdot 2 \neq 0 \implies \mathbf{n} \not\perp \mathbf{d} \implies l \nparallel \mathcal{P}$$

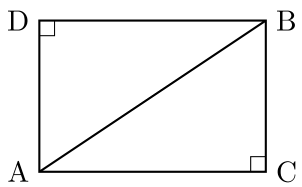
Assume $\mathbf{n} \parallel \mathbf{d}$. Let $c \in \mathbb{R}$ such that $\mathbf{n} = c\mathbf{d} \implies 2 = c \cdot 1$ and $5 = c \cdot (-3) \implies \nexists c \implies l \not\perp \mathcal{P}$

$$\therefore l \nparallel \mathcal{P} \text{ and } l \not\perp \mathcal{P}$$

Exercise 4 (0 points).

Prove, that the distance $d(B, l)$ from the point $B = (x_0, y_0)$ to the line l with general equation $ax + by = d$ is given by the formula $d(B, l) = \frac{|ax_0 + by_0 - d|}{\sqrt{a^2+b^2}}$.

Take an arbitrary point $A = (x, y)$ on the line l. Take points C on the line l and D, such that $AC \perp BC$,$AD \parallel \mathbf{n}$, where $\mathbf{n}$ is the normal vector of the line.



Distance between point B and line l is $BC = AD = \left\| \text{proj}_{\mathbf{n}} \vec{AB} \right\|$, by equality of triangles. By the general equation of the line we find that $\mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\vec{AB} = \begin{bmatrix} x_0 - x \\ y_0 - y \end{bmatrix}$.

$$\left\| \text{proj}_{\mathbf{n}} \vec{AB} \right\| = \left\| \frac{\mathbf{n} \cdot \vec{AB}}{\mathbf{n} \cdot \mathbf{n}} \mathbf{n} \right\| = \left\| \frac{a \cdot (x_0 - x) + b \cdot (y_0 - y)}{a \cdot a + b \cdot b} \begin{bmatrix} a \\ b \end{bmatrix} \right\| = \frac{|ax_0 + by_0 - (ax + by)|}{a^2 + b^2} \sqrt{a^2 + b^2}$$

$$\because ax + by = d \implies d(B, l) = \left\| \text{proj}_{\mathbf{n}} \vec{AB} \right\| = \frac{|ax_0 + by_0 - d|}{\sqrt{a^2 + b^2}}$$