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CS104 Linear Algebra Section D - Homework 3

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Exercise 1 (40 points).

The following complex numbers are given:

$$x = 2 - 2i, y = -2i, z = 2\sqrt{3} + 2i$$

(a) Compute $x + y$, $y - z$,$x \cdot z$, z/y,x/z.

$$x + y = 2 - 2i - 2i = 2 - 4i$$

$$y - z = -2i - 2\sqrt{3} - 2i = -2\sqrt{3} - 4i$$

$$x \cdot z = (2 - 2i)(2\sqrt{3} + 2i) = (4\sqrt{3} + 4) + i(4 - 4\sqrt{3})$$

$$\frac{z}{y} = \frac{2\sqrt{3} + 2i}{-2i} = \frac{(2\sqrt{3} + 2i) \cdot (2i)}{(-2i) \cdot (2i)} = \frac{-4 + 4\sqrt{3}i}{4} = -1 + \sqrt{3}i$$

$$\frac{x}{z} = \frac{2 - 2i}{2\sqrt{3} + 2i} = \frac{(2 - 2i) \cdot (2\sqrt{3} - 2i)}{(2\sqrt{3} + 2i) \cdot (2\sqrt{3} - 2i)} = \frac{(4\sqrt{3} - 4) + i(-4\sqrt{3} - 4)}{12 + 4} = \left(\frac{\sqrt{3}}{4} - \frac{1}{4}\right) + i\left(-\frac{\sqrt{3}}{4} - \frac{1}{4}\right)$$

(b) Write polar forms of x, y and z. Compute x^{20} and z^{30}.

$$r = \sqrt{2^2 + (-2)^2} = 2\sqrt{2} \text{ and } \theta = \arctan\left(\frac{-2}{2}\right) = -\frac{\pi}{4}$$

$$x = r(\cos\theta + i\sin\theta) = 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

$$r = \sqrt{2^2} = 2 \text{ and } \theta = -\frac{\pi}{2}$$

$$y = r(\cos\theta + i\sin\theta) = 2\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$$

$$r = \sqrt{(2\sqrt{3})^2 + 2^2} = 4 \text{ and } \theta = \arctan\left(\frac{2}{2\sqrt{3}}\right) = \frac{\pi}{6}$$

$$z = r(\cos\theta + i\sin\theta) = 4\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right) ss$$

$$x^n = r^n(\cos n\theta + i\sin n\theta) \implies x^{20} = (2\sqrt{2})^{20}(\cos(-5\pi) + i\sin(-5\pi)) = -8^{10}$$

$$z^n = r^n(\cos n\theta + i\sin n\theta) \implies z^{30} = 4^{30}(\cos(5\pi) + i\sin(5\pi)) = -4^{30}$$

(c) Find all complex numbers s such that $s^4 = x$.

$$s_k = r^{\frac{1}{n}} \left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i \sin\left(\frac{\theta + 2\pi k}{n}\right) \right), n = 4, k \in \{0, 1, 2, 3\}$$

$$s_0 = r^{\frac{1}{4}} \left(\cos\left(\frac{\theta}{4}\right) + i \sin\left(\frac{\theta}{4}\right) \right)$$

$$s_1 = r^{\frac{1}{4}} \left(\cos\left(\frac{\theta + 2\pi}{4}\right) + i \sin\left(\frac{\theta + 2\pi}{4}\right) \right)$$

$$s_2 = r^{\frac{1}{4}} \left(\cos\left(\frac{\theta + 4\pi}{4}\right) + i \sin\left(\frac{\theta + 4\pi}{4}\right) \right)$$

$$s_3 = r^{\frac{1}{4}} \left(\cos\left(\frac{\theta + 6\pi}{4}\right) + i \sin\left(\frac{\theta + 6\pi}{4}\right) \right)$$

Exercise 2 (20 points).

Consider the following complex vectors.

$$\mathbf{u} = \begin{bmatrix} -2i \\ -3 + 2i \\ 1 - 5i \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 2 \\ 2 - i \\ -2i \end{bmatrix}$$

(a) Calculate $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{u}$.

$$\mathbf{u} \cdot \mathbf{v} = \bar{u_1} v_1 + \bar{u_2} v_2 + \bar{u_3} v_3 = (2i)(2) + (-3 - 2i)(2 - i) + (1 + 5i)(-2i) = 2 + i$$

$$\mathbf{v} \cdot \mathbf{u} = \bar{v_1} u_1 + \bar{v_2} u_2 + \bar{v_3} u_3 = (2)(-2i) + (2 + i)(-3 + 2i) + (2i)(1 - 5i) = 2 - i$$

(b) Calculate $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$.

$$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{(-2i)(2i) + (-3 + 2i)(-3 - 2i) + (1 - 5i)(1 + 5i)}$$

$$= \sqrt{0^2 - (2i)^2 + (-3)^2 - (2i)^2 + (1)^2 - (5i)^2} = \sqrt{4 + 9 + 4 + 1 + 25} = \sqrt{43}$$

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{2 \cdot 2 + (2 - i)(2 + i) + (-2i)(2i)}$$

$$= \sqrt{2^2 + 2^2 - i^2 - 4i^2} = \sqrt{4 + 4 + 1 + 4} = \sqrt{13}$$

Exercise 3 (20 points).

Let $a = 7$, $b = 5$ and $c = 9$ be numbers in the field $\mathbb{Z}_{11}$. Compute $\frac{\frac{a}{b} + b^2}{c}$.

$$b \cdot b^{-1} = 1 \implies b^{-1} = 9, \text{ by trial and error}$$

$$c \cdot c^{-1} = 1 \implies c^{-1} = 5, \text{ by trial and error}$$

$$\frac{\frac{a}{b} + b^2}{c} = (a \cdot b^{-1} + b^2) \cdot c^{-1} = (7 \cdot 9 + 5^2) \cdot 5 = (8 + 3) \cdot 5 = 0$$

Exercise 4 (20 points).

Consider the following vectors in $\mathbb{Z}_7^4$.

$$\mathbf{u} = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 8 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 5 \end{bmatrix}$$

Compute $\mathbf{u}+\mathbf{v}$, $\mathbf{u}-\mathbf{v}$, $5\mathbf{u}$ and $4^{-1}\mathbf{v}$.

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} 1+2 \\ 5+4 \\ 0+6 \\ 8+5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 6 \\ 6 \end{bmatrix}$$

$$\mathbf{u}-\mathbf{v} = \begin{bmatrix} 1-2 \\ 5-4 \\ 0-6 \\ 8-5 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

$$5\mathbf{u} = \begin{bmatrix} 5\cdot 1 \\ 5\cdot 5 \\ 5\cdot 0 \\ 5\cdot 8 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 0 \\ 5 \end{bmatrix}$$

$$4^{-1} = 2 \implies 4^{-1}\mathbf{v} = 2\mathbf{v} = \begin{bmatrix} 2\cdot 2 \\ 2\cdot 4 \\ 2\cdot 6 \\ 2\cdot 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 5 \\ 3 \end{bmatrix}$$