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CS104 Linear Algebra Section D - Homework 4

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Exercise 1 (30 points).

Consider the following system of linear equations over $\mathbb{R}$.

$$\begin{cases} \frac{1}{2}x + y - z - 6w = 2 \\ \frac{1}{6}x + \frac{1}{2}y - 3w + v = -1 \\ \frac{1}{3}x - 2z - 4v = 8 \end{cases}$$

a) Bring the augmented matrix of this system to a row-echelon form (indicate each elementary operation you use).
Convert the system of equations to an augmented matrix.

$$\left[\begin{array}{ccccc|c} 1/2 & 1 & -1 & -6 & 0 & 2 \\ 1/6 & 1/2 & 0 & -3 & 1 & -1 \\ 1/3 & 0 & -2 & 0 & -4 & 8 \end{array}\right] \xleftrightarrow{\substack{2R_1 \\ 6R_2 \\ 3R_3}} \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -12 & 0 & 4 \\ 1 & 3 & 0 & -18 & 6 & -6 \\ 1 & 0 & -6 & 0 & -12 & 24 \end{array}\right] \xleftrightarrow{\substack{R_2-R_1 \\ R_3-R_1}}$$

$$\longleftrightarrow \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -12 & 0 & 4 \\ 0 & 1 & 2 & -6 & 6 & -10 \\ 0 & -2 & -8 & 12 & -12 & 20 \end{array}\right] \xleftrightarrow{R_3+2R_2} \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -12 & 0 & 4 \\ 0 & 1 & 2 & -6 & 6 & -10 \\ 0 & 0 & -4 & 0 & 0 & 0 \end{array}\right] \xleftrightarrow{-\frac{1}{4}R_3}$$

$$\longleftrightarrow \left[\begin{array}{ccccc|c} 1 & 2 & -2 & -12 & 0 & 4 \\ 0 & 1 & 2 & -6 & 6 & -10 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

b) Find the rank of the coefficients matrix and the rank of the augmented matrix of the system, decide if the system is consistent, then list its leading variables and free variables.
The rank of the augment matrix in row-echelon form is 3. So is the matrix of coefficients, hence this system is consistent. The system of equations has 5 variables, which is more than the rank of the augmented matrix. The leading variables are x, y and z, while w and v are the free variables.

c) Bring augmented matrix to its reduced row-echelon form (indicate each elementary operation you use). Solve the system by the Gauss-Jordan method.

$$\left[\begin{array}{ccccc|c} 1 & 2 & -2 & -12 & 0 & 4 \\ 0 & 1 & 2 & -6 & 6 & -10 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \xleftrightarrow{R_1-2R_2} \left[\begin{array}{ccccc|c} 1 & 0 & -6 & 0 & -12 & 24 \\ 0 & 1 & 2 & -6 & 6 & -10 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \xleftrightarrow{\substack{R_1+6R_3 \\ R_2-2R_3}}$$

$$\longleftrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -12 & 24 \\ 0 & 1 & 0 & -6 & 6 & -10 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \Longrightarrow \begin{cases} x-12v=24 \\ y-6w+6v=-10 \\ z=0 \end{cases} \Longrightarrow$$

$$\Longrightarrow \begin{cases} x=24+12v \\ y=-10+6w-6v \\ z=0 \end{cases} \Longrightarrow \text{Solution set} = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \\ v \end{bmatrix} \in \mathbb{R}^5, \text{s.t. } \begin{bmatrix} x \\ y \\ z \\ w \\ v \end{bmatrix} = \begin{bmatrix} 24+12s \\ -10+6t-6s \\ 0 \\ s \\ t \end{bmatrix}, s,t \in \mathbb{R} \right\}$$

Exercise 2 (10 points).

Solve the given system of equations on the field $\mathbb{R}$ by the Gaussian elimination:

$$\begin{cases} 8x-24y+22z=-7 \\ 2x-6y+7z=-\frac{13}{4} \\ 3z=-3 \end{cases}$$

Convert the system of equations to an augmented matrix.

$$\left[\begin{array}{ccc|c} 8 & -24 & 22 & -7 \\ 2 & -6 & 7 & -\frac{13}{4} \\ 0 & 0 & 3 & -3 \end{array}\right] \xleftrightarrow{R_2-\frac{1}{4}R_1} \left[\begin{array}{ccc|c} 8 & -24 & 22 & -7 \\ 0 & 6 & -37 & 0.25 \\ 0 & 0 & 3 & -3 \end{array}\right]$$

$$\begin{cases} 8x-24y+22z=-7 \\ 6y-37z=0.25 \\ 3z=-3 \end{cases} \Longrightarrow \begin{cases} z=-1 \\ 8x-24y-22=-7 \\ 6y+37=0.25 \end{cases} \Longrightarrow \begin{cases} z=-1 \\ y=-6.125 \\ 8x+147-22=-7 \end{cases} \Longrightarrow \begin{cases} z=-1 \\ y=-6.125 \\ x=-16.5 \end{cases}$$

$$\text{Solution set } = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3, \text{s.t. } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -16.5 \\ -6.125 \\ -1 \end{bmatrix} \right\}$$

Exercise 3 (15 points).

Find the line of intersection of the planes $2x+3y+3z=4$ and $3x+2y-2z=5$ by solving the appropriate system.
To find the intersection, we can solve fore the two equations of the planes, finding the set of points that satisfy both equations. Convert the resulting system of equations to an augmented matrix.

$$\begin{cases} 2x+3y+3z=4 \\ 3x+2y-2z=5 \end{cases} \Longrightarrow \left[\begin{array}{ccc|c} 2 & 3 & 3 & 4 \\ 3 & 2 & -2 & 5 \end{array}\right] \xleftrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 3 & 2 & -2 & 5 \\ 2 & 3 & 3 & 4 \end{array}\right] \xleftrightarrow{R_1-R_2}$$

$$\longleftrightarrow \left[\begin{array}{ccc|c} 1 & -1 & -5 & 1 \\ 2 & 3 & 3 & 4 \end{array}\right] \xleftrightarrow{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & -1 & -5 & 1 \\ 0 & 5 & 13 & 2 \end{array}\right] \xleftrightarrow{\frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & -1 & -5 & 1 \\ 0 & 1 & \frac{13}{5} & \frac{2}{5} \end{array}\right] \xleftrightarrow{R_1+R_2}$$

$$\longleftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -\frac{12}{5} & \frac{7}{5} \\ 0 & 1 & \frac{13}{5} & \frac{2}{5} \end{array}\right] \Longrightarrow \begin{cases} x-\frac{12}{5}z=\frac{7}{5} \\ y+\frac{13}{5}z=\frac{2}{5} \end{cases} \Longrightarrow \begin{cases} x=\frac{7}{5}+\frac{12}{5}z \\ y=\frac{2}{5}-\frac{13}{5}z \end{cases}$$

$$\text{line } l = \begin{bmatrix} 7/5+(12/5)z \\ 2/5-(13/5)z \\ z \end{bmatrix} = \begin{bmatrix} 7/5 \\ 2/5 \\ 0 \end{bmatrix} + t \begin{bmatrix} 12 \\ -13 \\ 5 \end{bmatrix}, t \in \mathbb{R}$$

Exercise 4 (15 points).

Determine whether the lines $\mathbf{x} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, t \in \mathbb{R}$ and $\mathbf{x} = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} + s \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, s \in \mathbb{R}$ intersect and, if so, find their point of intersection.

Writing the equations of the lines in terms of t and setting them to be equal to each other, we get a system of equations.

$$\begin{cases} 2 + 2t = 6 + 2s \\ 2 + t = 5 + 2s \\ 1 + 2t = 4 + s \end{cases} \implies \begin{cases} 2t - 2s = 4 \\ t - 2s = 3 \\ 2t - s = 3 \end{cases}$$

Now we can solve the system of equations by Gaussian elimination, by solving the augmented matrix of the system.

$$\left[\begin{array}{cc|c} 2 & -2 & 4 \\ 1 & -2 & 3 \\ 2 & -1 & 3 \end{array}\right] \xleftrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 2 & -2 & 4 \\ 2 & -1 & 3 \end{array}\right] \xleftrightarrow[R_3 - 2R_1]{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 2 & -2 \\ 0 & 3 & -3 \end{array}\right] \xleftrightarrow[\frac{1}{3}R_3]{\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{array}\right] \implies$$

$$\implies \begin{cases} t - 2s = 3 \\ s = -1 \end{cases} \implies \begin{cases} t = 1 \\ s = -1 \end{cases} \implies \vec{OP} = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix} \implies P = (4, 3, 3)$$

Exercise 5 (15 points).

Solve the following system of linear equations over $\mathbb{Z}_5$ by the Gauss Jordan elimination and define how many solutions does the system have?

$$\begin{cases} 2x_2 + 4x_3 + x_4 = 0 \\ x_1 + x_2 + 3x_3 + 3x_4 = 1 \\ 2x_1 + 3x_2 + 4x_3 + 3x_4 = 4 \\ 4x_2 = 4 \end{cases}$$

Convert the system of equations to an augmented matrix.

$$\left[\begin{array}{cccc|c} 0 & 2 & 4 & 1 & 0 \\ 1 & 1 & 3 & 3 & 1 \\ 2 & 3 & 4 & 3 & 4 \\ 0 & 4 & 0 & 0 & 4 \end{array}\right] \xleftrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cccc|c} 1 & 1 & 3 & 3 & 1 \\ 0 & 2 & 4 & 1 & 0 \\ 2 & 3 & 4 & 3 & 4 \\ 0 & 4 & 0 & 0 & 4 \end{array}\right] \xleftrightarrow{R_3 - 2R_1} \left[\begin{array}{cccc|c} 1 & 1 & 3 & 3 & 1 \\ 0 & 2 & 4 & 1 & 0 \\ 0 & 1 & 3 & 2 & 2 \\ 0 & 4 & 0 & 0 & 4 \end{array}\right] \xleftrightarrow{R_2 \leftrightarrow R_3}$$

$$\longleftrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 3 & 3 & 1 \\ 0 & 1 & 3 & 2 & 2 \\ 0 & 2 & 4 & 1 & 0 \\ 0 & 4 & 0 & 0 & 4 \end{array}\right] \xleftrightarrow[R_4 - 4R_2]{R_3 - 2R_2} \left[\begin{array}{cccc|c} 1 & 1 & 3 & 3 & 1 \\ 0 & 1 & 3 & 2 & 2 \\ 0 & 0 & 3 & 2 & 1 \\ 0 & 0 & 3 & 2 & 1 \end{array}\right] \xleftrightarrow[2R_3]{R_4 - R_3} \left[\begin{array}{cccc|c} 1 & 1 & 3 & 3 & 1 \\ 0 & 1 & 3 & 2 & 2 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xleftrightarrow[R_1 - 3R_3]{R_2 - 3R_3}$$

$$\longleftrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xleftrightarrow{R_1 - R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Rank of this augmented matrix is 3, number of variables is 4. x_1, x_2 and x_3 are leading variables, while x_4 is a free variable and $x_4 \in \mathbb{Z}_5$, meaning it can take five values. Hence the system has 5 solution.

$$\begin{cases} x_1 + x_4 = 4 \\ x_2 = 1 \\ x_3 + 4x_4 = 2 \end{cases} \implies \begin{cases} x_1 = 4 - x_4 \\ x_2 = 1 \\ x_3 = 2 - 4x_4 \end{cases} \quad \text{Solution set} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{Z}_5^4, \text{s.t.} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4-s \\ 1 \\ 2-4s \\ s \end{bmatrix}, s \in \mathbb{Z}_5 \right\}$$

Exercise 6 (15 points).

Solve the given system of equations on the complex field $\mathbb{C}$ using Gauss-Jordan elimination.

$$\begin{cases} -ix + (1-i)y = -3i \\ (2-2i)x + 6y = 10 - 6i \end{cases}$$

Convert the system of equations to an augmented matrix.

$$\left[\begin{array}{cc|c} -i & (1-i) & -3i \\ (2-2i) & 6 & (10-6i) \end{array}\right] \xleftrightarrow{iR_1} \left[\begin{array}{cc|c} 1 & 1+i & 3 \\ (2-2i) & 6 & (10-6i) \end{array}\right] \xleftrightarrow{R_2-(2-2i)R_2}$$

$$\longleftrightarrow \left[\begin{array}{cc|c} 1 & 1+i & 3 \\ 0 & 2 & 4 \end{array}\right] \xleftrightarrow{\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 1+i & 3 \\ 0 & 1 & 2 \end{array}\right] \xleftrightarrow{R_1-(1+i)R_2} \left[\begin{array}{cc|c} 1 & 0 & 1-2i \\ 0 & 1 & 2 \end{array}\right] \implies$$

$$\implies \begin{cases} x = 1 - 2i \\ y = 2 \end{cases} \implies \text{Solution set} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{C}^2, \text{s.t.} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1-2i \\ 2 \end{bmatrix} \right\}$$