Digital version

CS104 Linear Algebra Section D - Homework 5

Mher Saribekyan A09210183

February 28, 2024

Exercise 1 (5 points).

Let $A = \begin{bmatrix} 1 & 4 & -8 \\ -1 & 3 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1-2i & 2+2i \\ i & 3-i \\ 3-2i & -i \end{bmatrix}$. Compute $2A + iC^T$

$$2A + iC^T = 2\begin{bmatrix} 1 & 4 & -8 \\ -1 & 3 & 0 \end{bmatrix} + i\begin{bmatrix} 1-2i & 2+2i \\ i & 3-i \\ 3-2i & -i \end{bmatrix}^T = \begin{bmatrix} 2 & 8 & -16 \\ -2 & 6 & 0 \end{bmatrix} + i\begin{bmatrix} 1-2i & i & 3-2i \\ 2+2i & 3-i & -i \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 8 & -16 \\ -2 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 2+i & -1 & 2+3i \\ -2+2i & 1+3i & 1 \end{bmatrix} = \begin{bmatrix} 4+i & 7 & -14+3i \\ -4+2i & 7+3i & 1 \end{bmatrix}$$

Exercise 2 (5 points).

Let $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $D = [2, -1, 1]$, $E = \begin{bmatrix} 2+i & 1+i \\ 2i & 2-i \end{bmatrix}$.

For each item, decide whether or not the given expression is defined. For each item that is defined, compute the result:

a) $AB = \begin{bmatrix} 2 & -4 & 6 \\ 2 & 2 & 0 \end{bmatrix}$

b) CA is not defined.

c) $A^T E = \begin{bmatrix} 2+5i & 5-i \\ 2+i & 1+i \\ -4+6i & 6-6i \end{bmatrix}$

d) $BD^T = \begin{bmatrix} 5 \\ 5 \\ -2 \end{bmatrix}$

e) $A^T A = \begin{bmatrix} 5 & 1 & 6 \\ 1 & 1 & -2 \\ 6 & -2 & 20 \end{bmatrix}$

Exercise 3 (10 points).

Compute the following product using the indicated partitioning.

$$\left[\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 0 & 0 & 2 & 1 \end{array}\right] \cdot \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \cdot \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11} \cdot B_{11} + A_{12}B_{21} & A_{11} \cdot B_{12} + A_{12}B_{22} \\ A_{21} \cdot B_{11} + A_{22}B_{21} & A_{21} \cdot B_{12} + A_{22}B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 \end{bmatrix} \end{bmatrix} = \left[\begin{array}{cc|cc} 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 4 & 1 & 2 & 1 \end{array}\right]$$

Exercise 4 (10 points).

a) Prove that the main diagonal of a skew-symmetric matrix must consist entirely of zeros.
Let A be an $n \times n$ skew-symmetric matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, A = -A^T \implies A + A^T = 0$$

$$A+A^T = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{21} & \dots & a_{n1} \\ a_{12} & a_{22} & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}+a_{11} & a_{12}+a_{21} & \dots & a_{1n}+a_{n1} \\ a_{21}+a_{12} & a_{22}+a_{22} & \dots & a_{2n}+a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{n1}+a_{1n} & a_{n2}+a_{2n} & \dots & a_{nn}+a_{nn} \end{bmatrix} = 0$$

$$2a_{11} = 0, 2a_{22} = 0, ..., 2a_{nn} = 0 \implies \text{Main diagonal must constist entirely of zeros.}$$

Another property we see is $a_{ij} + a_{ji} = 0$, which implies that $a_{ij} = -a_{ji}$, $i, j \in \{1, 2, ..., n\}$

b) If A and B are skew-symmetric 2×2 matrices, under what conditions is AB skewsymmetric?

$$A = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \implies AB = \begin{bmatrix} -ab & 0 \\ 0 & -ab \end{bmatrix}$$

AB is a skew-symmetric matrix, if $a = 0$ or $b = 0$, hence one of the matrices A or B have to be the zero matrix, in which case AB will also be the zero matrix.

Exercise 5 (10 points).

Solve the given matrix equation for X. Simplify your answers as much as possible. Assume that all matrices are invertible:

a) $A^3X = A^{-2}$

$$A^3X = A^{-2} \overset{A^{-n}=(A^{-1})^n}{\implies} A^{-1}A^{-1}A^{-1}AAAX = A^{-1}A^{-1}A^{-1}(A^{-1})^2 \implies$$

$$\overset{AA^{-1}=I}{\implies} X = A^{-1}A^{-1}A^{-1}A^{-1}A^{-1} \implies X = A^{-5}$$

b) $(B^{-3}X)^{-1} = A(B^{-2}A)^{-1}$

$$(B^{-3}X)^{-1} = A(B^{-2}A)^{-1} \overset{B^{-n}=(B^{-1})^n}{\Longrightarrow} ((B^{-1})^3X)^{-1} = A((B^{-1})^2A)^{-1} \Longrightarrow$$

$$\Longrightarrow (B^{-1}B^{-1}B^{-1}X)^{-1} = A(B^{-1}B^{-1}A)^{-1} \Longrightarrow X^{-1}BBB = AA^{-1}BB \overset{AA^{-1}=I}{\Longrightarrow}$$
$$\Longrightarrow X^{-1}BBBB^{-1}B^{-1} = BBB^{-1}B^{-1} \Longrightarrow X^{-1}B = I \Longrightarrow X = B$$

Exercise 6 (20 points).

Find a sequence of elementary matrices E_1, E_2, ..., E_k such that $E_k...E_2E_1A = I$ or explain why it is impossible. If such sequence exists, use it to write both A and A^{-1} as a product of elementary matrices.

a) $A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$ Construct the augmented matrix $[A|I]$ and reduce to $[I|A^{-1}]$ with elementary operations.

$$\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \overset{R_1 \longleftrightarrow R_2}{\longrightarrow} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 2 & 4 & 1 & 0 \end{array}\right] \overset{R_2-2R_1}{\longrightarrow} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & -2 \end{array}\right] \longrightarrow$$

$$\overset{\frac{1}{2}R_2}{\longrightarrow} \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 1 & \frac{1}{2} & -1 \end{array}\right] \overset{R_1-R_2}{\longrightarrow} \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{2} & 2 \\ 0 & 1 & \frac{1}{2} & -1 \end{array}\right] \Longrightarrow A^{-1} = \begin{bmatrix} -\frac{1}{2} & 2 \\ \frac{1}{2} & -1 \end{bmatrix}$$

$$E_4E_3E_2E_1A = I, E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, E_4 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$E_1^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, E_4^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1} = E_4E_3E_2E_1I \text{ and } A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}I$$

b) $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 6 & 0 & 1 \end{array}\right] \overset{R_2-2R_1}{\longrightarrow} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right]$$

This matrix cannot be brought to the identity matrix I_2 with elementary operations, which means it does not have an inverse. We also know that, because the determinant of the matrix is $1 \cdot 6 - 2 \cdot 3 = 0$. Which is why there does not exist a sequence of elementary matrices such that $E_k...E_2E_1A = I$.

Exercise 7 (30 points).

Consider the matrix $B = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & -4 \\ -2 & 0 & 6 \end{bmatrix}$

a) Find the inverse of B by the Gauss-Jordan method.

$$\left[\begin{array}{ccc|ccc} 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & -4 & 0 & 1 & 0 \\ -2 & 0 & 6 & 0 & 0 & 1 \end{array}\right] \overset{R_1 \longleftrightarrow R_2}{\longrightarrow} \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ -2 & 0 & 6 & 0 & 0 & 1 \end{array}\right] \longrightarrow$$

$$\xrightarrow{R_3+2R_1}\left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 2 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -2 & 0 & 2 & 1 \end{array}\right] \longrightarrow$$

$$\xrightarrow{-\frac{1}{2}R_3}\left[\begin{array}{ccc|ccc} 1 & 0 & -4 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -\frac{1}{2} \end{array}\right] \xrightarrow{R_1+4R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -3 & 2 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -\frac{1}{2} \end{array}\right]$$

$$B^{-1} = \begin{bmatrix} 0 & -3 & 2 \\ \frac{1}{2} & 0 & 0 \\ 0 & -1 & -\frac{1}{2} \end{bmatrix}$$

b) Use the elimination process done in point (a) to represent B as a product of elementary matrices.

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_1^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}, E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$E_5 = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_5^{-1} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}I$$

c) Find the number of solutions of the system of linear equations $Ax = \mathbf{b}$, where $A = 2B^4$ and $\mathbf{b}$ is a vector in $\mathbb{R}^3$.

$$A = 2B^4 \implies A^{-1} = (2B^4)^{-1} = \frac{1}{2}(B^4)^{-1} = \frac{1}{2}(B^{-1})^4 \implies A^{-1} \text{ exists}$$

$$A \text{ is invertible } \implies Ax = \mathbf{b} \implies A^{-1}Ax = A^{-1}\mathbf{b} \implies x = A^{-1}\mathbf{b} \implies \text{equation has one solution}$$

d) Use the inverse B^{-1} obtained in point (a) to find the inverse of the matrix $3B^T$.

$$(3B^T)^{-1} = \frac{1}{3}(B^T)^{-1} = \frac{1}{3}(B^{-1})^T = \frac{1}{3}\begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -3 & 0 & -1 \\ 2 & 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{6} & 0 \\ -1 & 0 & -\frac{1}{3} \\ \frac{2}{3} & 0 & -\frac{1}{6} \end{bmatrix}$$

Exercise 8 (10 points).

Use the Gauss-Jordan method to check if the given matrix is invertible or not, and if it is invertible, find its inverse.

a) $D = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 7 & 0 \\ 1 & 0 & 5 \end{bmatrix}$ over $\mathbb{R}$

$$\left[\begin{array}{ccc|ccc} 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 7 & 0 & 0 & 1 & 0 \\ 1 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 - \frac{2}{7}R_2} \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & -\frac{2}{7} & 0 \\ 0 & 7 & 0 & 0 & 1 & 0 \\ 1 & 0 & 5 & 0 & 0 & 1 \end{array}\right]$$

This matrix cannot be brought to the identity matrix I_3 with elementary operations, which means it does not have an inverse.

b) $E = \begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}$ over $\mathbb{Z}_7$

$$\left[\begin{array}{cc|cc} 5 & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \longleftrightarrow R_2} \left[\begin{array}{cc|cc} 2 & 3 & 0 & 1 \\ 5 & 2 & 1 & 0 \end{array}\right] \xrightarrow{4R_1} \left[\begin{array}{cc|cc} 1 & 5 & 0 & 4 \\ 5 & 2 & 1 & 0 \end{array}\right] \xrightarrow{R_2 + 2R_1} \left[\begin{array}{cc|cc} 1 & 5 & 0 & 4 \\ 0 & 5 & 1 & 1 \end{array}\right] \longrightarrow$$

$$\xrightarrow{3R_2} \left[\begin{array}{cc|cc} 1 & 5 & 0 & 4 \\ 0 & 1 & 3 & 3 \end{array}\right] \xrightarrow{R_1 + 2R_2} \left[\begin{array}{cc|cc} 1 & 0 & 6 & 3 \\ 0 & 1 & 3 & 3 \end{array}\right] \Longrightarrow E^{-1} = \begin{bmatrix} 6 & 3 \\ 3 & 3 \end{bmatrix}$$