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CS104 Linear Algebra Section D - Homework 6

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Exercise 1 (20 points).

In each case below check if the given set V is a vector space over the given field $\mathbb{F}$ with the specified operations of addition and multiplication by a scalar. If it is not, list all of the axioms that fail to hold.

a) $V = \mathbb{R}^2$, $\mathbb{F} = \mathbb{R}$, with the usual addition, but the scalar multiplication defined in the following way:

$$\text{for } \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2, \text{ and } c \in \mathbb{R}, c \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ cy \end{bmatrix} \in \mathbb{R}^2$$

$$\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix} \in V, s, t \in \mathbb{R}$$

$$(r+s) \cdot \mathbf{v} = (r+s) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ (r+s)y \end{bmatrix} = \begin{bmatrix} x \\ r \cdot y \end{bmatrix} + \begin{bmatrix} 0 \\ s \cdot y \end{bmatrix} \neq r\mathbf{v} + s\mathbf{v}$$

$$\text{Distributive property does not hold(Axiom 10)} \implies V \text{ is not a vector space}$$

b) $V = \mathbb{R}^2$, $\mathbb{F} = \mathbb{C}$, with the usual scalar multiplication, but the addition defined in the following way:

$$\text{for } \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in \mathbb{C}^2, \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + 1 \\ y_1 + y_2 + 1 \end{bmatrix} \in \mathbb{C}^2$$

$$\mathbf{u} = \begin{bmatrix} a \\ b \end{bmatrix}, \mathbf{v} = \begin{bmatrix} c \\ d \end{bmatrix} \in \mathbb{R}^2, s, t \in \mathbb{C}$$

$$\text{Axiom 2) } s := i, s \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ai \\ bi \end{bmatrix} \notin \mathbb{R}^2$$

$$\text{Axiom 9) } s(\mathbf{u} + \mathbf{v}) = s \begin{bmatrix} a + c + 1 \\ b + d + 1 \end{bmatrix} = \begin{bmatrix} as + cs + s \\ bs + ds + s \end{bmatrix} \neq \begin{bmatrix} as + cs + 1 \\ bs + ds + 1 \end{bmatrix} = s\mathbf{u} + s\mathbf{v}$$

$$\text{Axiom 10) } (r+s) \cdot \mathbf{u} = (r+s) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ra + sa \\ rb + sb \end{bmatrix} \neq \begin{bmatrix} ra + sa + 1 \\ rb + sb + 1 \end{bmatrix} = r\mathbf{u} + s\mathbf{u}$$

Closure under multiplication and distributive properties do not hold (Axioms 2,9 and 10)

$\therefore V$ is not a vector space

c) $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{Z}_5^3, \text{s.t.} 3x + 2z + 2 = 1 \right\}, F = \mathbb{Z}_5$, with the usual operation of vector addition and scalar multiplication.

$$3x + 2z + 2 = 1 \implies x = z + 3 \implies V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{Z}_5^3, \text{s.t. } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b+3 \\ a \\ b \end{bmatrix}, a, b \in \mathbb{Z}_5 \right\}$$

$$v_1, v_2 \in V, r, s \in \mathbb{Z}_5$$

$$\text{Axiom 1) } v_1 + v_2 = \begin{bmatrix} z_1 + 3 \\ y_1 \\ z_1 \end{bmatrix} + \begin{bmatrix} z_2 + 3 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_1 + z_2 + 1 \\ y_1 + y_2 \\ z_1 + z_2 \end{bmatrix} \notin V$$

$$\text{Axiom 2) } rv_1 := 2 \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \cdot 3 \\ 2 \cdot 0 \\ 2 \cdot 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \notin V$$

$$\therefore V \text{ is not a vector space}$$

d) $V = M_{2\times 2}(\mathbb{R})$, with the operation of addition defined by $A \bigoplus B = AB$, and the usual scalar multiplication.

$$A, B \in V, r, s \in \mathbb{R}$$

$$\text{Axiom 3) } A \bigoplus B = AB \neq BA = B \bigoplus A$$

$$\text{Axiom 9) } r \cdot (A \bigoplus B) = r \cdot AB \neq r \cdot A \cdot r \cdot B = (r \cdot A)(r \cdot B) = r \cdot A \bigoplus r \cdot B$$

$$\therefore V \text{ is not a vector space}$$

Exercise 2 (10 points).

Determine whether the given set S of vectors with the usual addition and scalar multiplication is a vector space. In each case, take the set of scalars to be the set of all real numbers.

a) The set $S := \{A \in M_{n\times n}(\mathbb{R}) : A \text{ is a lower triangular}\}$.

$$\text{We know that } V = M_{n\times n}(\mathbb{R}) \text{ is a vector space, and } S \subset V$$

$$A := \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, B := \begin{bmatrix} b_{11} & 0 & \dots & 0 \\ b_{21} & b_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix} \in S, c \in \mathbb{R}$$

$$\text{Axiom 1) } \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}, + \begin{bmatrix} b_{11} & 0 & \dots & 0 \\ b_{21} & b_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{bmatrix} \overset{\text{Ressign scalars}}{=} \begin{bmatrix} c_{11} & 0 & \dots & 0 \\ c_{21} & c_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} \in S$$

$$\text{Axiom 2) } cA = \begin{bmatrix} ca_{11} & 0 & \dots & 0 \\ ca_{21} & ca_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ ca_{n1} & ca_{n2} & \dots & ca_{nn} \end{bmatrix} \overset{\text{Ressign scalars}}{=} \begin{bmatrix} c_{11} & 0 & \dots & 0 \\ c_{21} & c_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ c_{n1} & c_{n2} & \dots & c_{nn} \end{bmatrix} \in S$$

$$\therefore S \text{ is a subspace of } V \implies S \text{ is a vector space}$$

b) The set $S := \{A \in M_{2\times 2}(R) : det(A) \neq 0\}$.

$$\text{Axiom 1) Take } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in V, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} =: C, det(C) = 0 \implies C \notin V$$

$$\therefore V \text{ is not a vector space}$$

Exercise 3 (20 points).

Express S in set notation and determine whether it is a subspace of the given vector space V.

a) $V = \mathbb{R}^3$ and S is the set of all vectors (x, y, z) in V such that $y = -2x$, $z = x + 2y$.

$$z = -3x \implies S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{s.t. } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} s \\ -2s \\ -3s \end{bmatrix}, s \in \mathbb{R} \right\} \implies S \in V$$

$$\text{Check closures for } S, \text{ take } \mathbf{v}_1 = \begin{bmatrix} p \\ -2p \\ -3p \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} q \\ -2q \\ -3q \end{bmatrix} \in S, p, q, c \in \mathbb{R}$$

$$\text{Axiom 1) } \mathbf{v}_1 + \mathbf{v}_2 = \begin{bmatrix} p \\ -2p \\ -3p \end{bmatrix} + \begin{bmatrix} q \\ -2q \\ -3q \end{bmatrix} = \begin{bmatrix} p+q \\ -2p-2q \\ -3p-3q \end{bmatrix} = \begin{bmatrix} (p+q) \\ -2(p+q) \\ -3(p+q) \end{bmatrix} \overset{p+q:=a}{=} \begin{bmatrix} a \\ -2a \\ -3a \end{bmatrix} \in S$$

$$\text{Axiom 2) } c\mathbf{v}_1 = c \begin{bmatrix} p \\ -2p \\ -3p \end{bmatrix} = \begin{bmatrix} cp \\ -2cp \\ -3cp \end{bmatrix} \overset{cp:=a}{=} \begin{bmatrix} a \\ -2a \\ -3a \end{bmatrix} \in S \implies S \text{ is a subspace of } V$$

b) $V = \mathbb{R}^4$and S is the set of all solutions to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$, where A is a fixed 3×4 matrix.

$$S = \left\{ \mathbf{x}, \text{ such that } A\mathbf{x} = \mathbf{0}, A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{32} & a_{33} & a_{34} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}, a_{ij} \in \mathbb{R}, i \in 1,2,3, j \in 1,2,3,4 \right\}$$

$$\text{Take } \mathbf{x}, \mathbf{y} \in S, c \in \mathbb{R}$$

$$\text{Axiom 1) } \begin{cases} A\mathbf{x} = \mathbf{0} \\ A\mathbf{y} = \mathbf{0} \end{cases} \implies A\mathbf{x} + A\mathbf{y} = \mathbf{0} \overset{\text{distributive property}}{\implies} A(\mathbf{x}+\mathbf{y}) = \mathbf{0} \implies \mathbf{x} + \mathbf{y} \in S$$

$$\text{Axiom 2) } A\mathbf{x} = \mathbf{0} \implies c(A\mathbf{x}) = c(\mathbf{0}) \overset{\text{property of matrices}}{\implies} A(c\mathbf{x}) = 0 \implies c\mathbf{x} \in S$$

$$\therefore S \text{ is a subspace of } V$$

c) $V = M_{3\times 3}(\mathbb{R})$ and S is the subset of all 3×3 symmetric matrices.

$$S = \left\{ \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}, a, b, c, d, e, f \in \mathbb{R} \right\}$$

$$\text{Take } A = \begin{bmatrix} a_A & b_A & c_A \\ b_A & d_A & e_A \\ c_A & e_A & f_A \end{bmatrix}, B = \begin{bmatrix} a_B & b_B & c_B \\ b_b & d_B & e_B \\ c_B & e_B & f_B \end{bmatrix}, c \in \mathbb{R}$$

$$\text{Axiom 1) } A + B = \begin{bmatrix} a_A & b_A & c_A \\ b_A & d_A & e_A \\ c_A & e_A & f_A \end{bmatrix} + \begin{bmatrix} a_B & b_B & c_B \\ b_B & d_B & e_B \\ c_B & e_B & f_B \end{bmatrix} = \begin{bmatrix} a_A + a_B & b_A + b_B & c_A + c_B \\ b_A + b_B & d_A + d_B & e_A + e_B \\ c_A + c_B & e_A + e_B & f_A + f_B \end{bmatrix}$$

$$\overset{\text{Assign new scalars}}{=} \begin{bmatrix} a_C & b_C & c_C \\ b_C & d_C & e_C \\ c_C & e_C & f_C \end{bmatrix} \in S \implies A + B \in S$$

$$\text{Axiom 2) } cA = \begin{bmatrix} c \cdot a_A & c \cdot b_A & c \cdot c_A \\ c \cdot b_A & c \cdot d_A & c \cdot e_A \\ c \cdot c_A & c \cdot e_A & c \cdot f_A \end{bmatrix} \overset{\text{Assign new scalars}}{=} \begin{bmatrix} a_C & b_C & c_C \\ b_C & d_C & e_C \\ c_C & e_C & f_C \end{bmatrix} \in S \implies S \text{ is a subspace of } V$$

d) V is a vector space of all real valued functions and S is the set of all solutions to the differential equation $2y' + y = 0$.

$$S = \{y, \text{ such that } 2y' + y = 0\}$$

$$\text{Check closures for } S, \text{ take } y, z \in S, c \in \mathbb{R}$$

$$\text{Axiom 1) } \begin{cases} 2y' + y = 0 \\ 2z' + z = 0 \end{cases} \implies 2y' + y + 2z' + z = 0 \overset{\text{Sum rule}}{\implies} 2(y+z)' + (y+z) = 0 \implies y + z \in S$$

$$\text{Axiom 2) } 2y' + y = 0 \implies c \cdot (2y' + y) = c \cdot 0 \overset{\text{Calculus magic}}{\implies} 2(cy)' + (cy) = 0 \implies cy \in S$$

$$\therefore S \text{ is a subspace of } V$$

Exercise 4 (20 points).

Show that $\mathbf{v}_1 = (2, -3, 1)$, $\mathbf{v}_2 = (2, 2, -5)$, $\mathbf{v}_3 = (1, 2, -4)$ span $\mathbb{R}^3$ and express $\mathbf{v} = (9, 8, 7)$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

$$\text{Check if } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \text{ are linearly independent in } \mathbb{R}^3$$

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}, \text{ for some values of } c_1, c_2, c_3 \in \mathbb{R}$$

$$\left[\begin{array}{ccc|c} 2 & 2 & 1 & 0 \\ -3 & 2 & 2 & 0 \\ 1 & -5 & -4 & 0 \end{array}\right] \overset{R_1 \longleftrightarrow R_3}{\longleftrightarrow} \left[\begin{array}{ccc|c} 1 & -5 & -4 & 0 \\ -3 & 2 & 2 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right] \overset{\substack{R_2 + 3R_1 \\ R_3 - 2R_1}}{\longleftrightarrow} \left[\begin{array}{ccc|c} 1 & -5 & -4 & 0 \\ 0 & -13 & -10 & 0 \\ 0 & 12 & 9 & 0 \end{array}\right] \longleftrightarrow$$

$$\overset{\substack{-\frac{1}{13}R_2 \\ R_3 - 12R_2}}{\longleftrightarrow} \left[\begin{array}{ccc|c} 1 & -5 & -4 & 0 \\ 0 & 1 & \frac{10}{13} & 0 \\ 0 & 0 & -\frac{3}{13} & 0 \end{array}\right] \overset{\substack{-\frac{13}{3}R_3 \\ R_1 + 4R_3 \\ R_2 - \frac{10}{13}R_3}}{\longleftrightarrow} \left[\begin{array}{ccc|c} 1 & -5 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \overset{R_1 + 5R_2}{\longleftrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \implies$$

$$\implies \begin{cases} c_1 = 0 \\ c_2 = 0 \\ c_3 = 0 \end{cases} \implies \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \text{ are linearly independent in } \mathbb{R}^3 \implies \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^3$$

$$b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3 = \mathbf{v}, \text{ for some values of } b_1, b_2, b_3 \in \mathbb{R}$$

$$
\left[\begin{array}{ccc|c} 2 & 2 & 1 & 9 \\ -3 & 2 & 2 & 8 \\ 1 & -5 & -4 & 7 \end{array}\right] \xleftrightarrow{R_1 \longleftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -5 & -4 & 7 \\ -3 & 2 & 2 & 8 \\ 2 & 2 & 1 & 9 \end{array}\right] \xleftrightarrow[R_3-2R_1]{R_2+3R_1} \left[\begin{array}{ccc|c} 1 & -5 & -4 & 7 \\ 0 & -13 & -10 & 29 \\ 0 & 12 & 9 & -5 \end{array}\right] \longleftrightarrow
$$

$$
\xleftrightarrow[R_3-12R_2]{-\frac{1}{13}R_2} \left[\begin{array}{ccc|c} 1 & -5 & -4 & 7 \\ 0 & 1 & \frac{10}{13} & -\frac{29}{13} \\ 0 & 0 & -\frac{3}{13} & \frac{283}{13} \end{array}\right] \xleftrightarrow[R_2-\frac{10}{13}R_3]{\substack{-\frac{13}{3}R_3 \\ R_1+4R_3}} \left[\begin{array}{ccc|c} 1 & -5 & 0 & -\frac{1111}{3} \\ 0 & 1 & 0 & \frac{211}{3} \\ 0 & 0 & 1 & -\frac{283}{3} \end{array}\right] \xleftrightarrow{R_1+5R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{56}{3} \\ 0 & 1 & 0 & \frac{211}{3} \\ 0 & 0 & 1 & -\frac{283}{3} \end{array}\right] \implies
$$

$$
\implies \begin{cases} b_1 = -\frac{56}{3} \\ b_2 = \frac{211}{3} \\ b_3 = -\frac{283}{3} \end{cases} \implies -\frac{56}{3}\mathbf{v}_1 + \frac{211}{3}\mathbf{v}_2 - \frac{283}{3}\mathbf{v}_3 = \mathbf{v}
$$

Exercise 5 (30 points).

a) Let S be the subspace of $M_3(\mathbb{R})$ consisting of all 3×3 skew-symmetric matrices. Find a set of vectors that spans S.

$$
\text{Let } A \text{ be a } 3\times 3 \text{ skew-symmetric matrix } \implies A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}, A \in S, \forall a, b, c \in \mathbb{R}
$$

$$
A = a\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ -0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}
$$

$$
S = \operatorname{span}\left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ -0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\}
$$

b) If $p_1(x) = x - 2$, and $p_2(x) = 2x^2 + 2x - 3$, determine whether $p(x) = 2x^2 + 4x - 7$ lies in $\operatorname{span}\{p_1, p_2\}$.

$$
\forall p_k(x) \in \operatorname{span}\{p_1, p_2\}, \exists c_1, c_2 \in \mathbb{R} \text{ such that } p_k(x) \equiv c_1 p_1(x) + c_2 p_2(x)
$$

$$
\text{Assume } p(x) \in \operatorname{span}\{p_1, p_2\} \implies \exists c_1, c_2 \in \mathbb{R}, \text{ s.t. } 2x^2 + 4x - 7 \equiv c_1(x-2) + c_2(2x^2 + 2x - 3)
$$

$$
2x^2 + 4x - 7 \equiv c_1 x - 2c_1 + 2x^2 c_2 + 2xc_2 - 3c_2
$$

$$
2x^2 + 4x + (-7) \equiv (2c_2)x^2 + (c_1 + 2c_2)x + (-2c_1 - 3c_2)1 \implies \begin{cases} c_1 = 2 \\ c_2 = 1 \end{cases}
$$

$$
\therefore p(x) = 2x^2 + 4x - 7 \in \operatorname{span}\{p_1, p_2\}
$$

c) Consider $A_1 = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 2 & 1 \\ 2 & -3 \end{bmatrix}$ in $M_2(\mathbb{R})$. Find $\operatorname{span}\{A_1, A_2\}$ and determine whether or not $B = \begin{bmatrix} 5 & 2 \\ 7 & -5 \end{bmatrix}$ lies in this subspace.

$$
\forall A_k \in \operatorname{span}\{A_1, A_2\} \implies \exists c_1, c_2 \in \mathbb{R} \text{ such that } A_k = c_1 A_1 + c_2 A_2
$$

$$
A_k = c_1\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} + c_2\begin{bmatrix} 2 & 1 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 1c_1 + 2c_2 & c_2 \\ 3c_1 + 2c_2 & 2c_1 - 3c_2 \end{bmatrix}
$$

$$\text{span}\{A_1, A_2\} = \left\{ \begin{bmatrix} 1s + 2t & t \\ 3s + 2t & 2s - 3t \end{bmatrix}, s, t \in \mathbb{R} \right\}$$

$$\text{Assume } B \in \text{span}\{A_1, A_2\} \implies \begin{bmatrix} 5 & 2 \\ 7 & -5 \end{bmatrix} = \begin{bmatrix} 1s + 2t & t \\ 3s + 2t & 2s - 3t \end{bmatrix} \implies \nexists s, t \in \mathbb{R}$$

$$\therefore B \notin \text{span}\{A_1, A_2\}$$