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CS104 Linear Algebra Section D - Homework 8

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Exercise 1 (10 points).

a) Extend $\{1+x; 1-2x\}$ to a basis for $P_2(\mathbb{R})$.

$$P_2(\mathbb{R}) = \text{span}\{1, x, x^2\}, \dim(P_2(\mathbb{R})) = 3$$

$$\{1+x; 1-2x, 1, x, x^2\}$$

$$1 = \frac{2}{3}(1+x) + \frac{1}{3}(1-2x)$$

$$x = \frac{1}{3}(1+x) - \frac{1}{3}(1-2x)$$

$$\text{span}\{1+x; 1-2x, x^2\} = P_2(\mathbb{R}) \text{ and they are linearly independent}$$

$$\therefore \{1+x; 1-2x, x^2\} \text{ is basis for } P_2(\mathbb{R})$$

b) Extend $\left\{\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right\}$ to be a basis for $M_2(\mathbb{R})$

$$M_2(\mathbb{R}) = \text{span}\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}, \dim(M_2(\mathbb{R})) = 4$$

$$\left\{\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$$

$$\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} - \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}$$

$$\text{span}\left\{\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right\} = M_2(\mathbb{R}) \text{ and they are linearly independent}$$

$$\therefore \left\{\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\right\} \text{ is basis for } M_2(\mathbb{R})$$

Exercise 2 (20 points).

Consider the following matrix A

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 & 0 \\ 5 & -15 & -2 & 1 & -4 \\ 2 & -6 & -1 & 0 & -1 \\ 0 & 0 & -2 & -4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & 2 & 0 \\ 5 & -15 & -2 & 1 & -4 \\ 2 & -6 & -1 & 0 & -1 \\ 0 & 0 & -2 & -4 & 5 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 1 & -3 & -\frac{2}{5} & \frac{1}{5} & -\frac{4}{5} \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & -\frac{1}{5} & -\frac{2}{5} & \frac{3}{5} \\ 0 & 0 & -2 & -4 & 5 \end{bmatrix} \longleftrightarrow \begin{bmatrix} 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) Determine weather[1] $\mathbf{b} = \begin{bmatrix} 0 & 1 & 3 & 0 \end{bmatrix}^T$ belongs to col(A).

$$\text{col}(A) = \text{span}\left\{ \begin{bmatrix} 0 \\ 5 \\ 2 \\ 0 \end{bmatrix} ; \begin{bmatrix} 1 \\ -2 \\ -1 \\ -2 \end{bmatrix} ; \begin{bmatrix} 0 \\ -4 \\ -1 \\ 5 \end{bmatrix} \right\}, \text{ it is clear that } \mathbf{b} \notin \text{col}(A)$$

b) Determine weather[2] $\mathbf{w} = \begin{bmatrix} 1 & -3 & 2 & 5 & -1 \end{bmatrix}$ belongs to row(A).

$$\text{row}(A) = \text{span}\{\begin{bmatrix} 1 & -3 & 0 & 1 & 0 \end{bmatrix} ; \begin{bmatrix} 0 & 0 & 1 & 2 & 0 \end{bmatrix} ; \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix}\}$$

$$\text{It is clear that } \mathbf{w} \in \text{row}(A)$$

c) Determine weather $\mathbf{x} = \begin{bmatrix} 4 & 1 & 2 & -1 & 0 \end{bmatrix}^T$ belongs to null(A).

$$A\mathbf{x} = \begin{bmatrix} 0 & 0 & 1 & 2 & 0 \\ 5 & -15 & -2 & 1 & -4 \\ 2 & -6 & -1 & 0 & -1 \\ 0 & 0 & -2 & -4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \mathbf{0} \therefore \mathbf{x} \in \text{null}(A)$$

d) Describe row(A) (find a basis for it).
Refer to point b)

e) Describe col(A) (find a basis for it).
Refer to point a)

f) Find rank(A), and compute nullity(A) using the rank theorem.

$$\text{rank}(A) = 3 \therefore \text{nullity}(A) = 4 - 3 = 1$$

g) Conclude if row(A) $= \mathbb{R}^5$ or col(A) $= \mathbb{R}^4$.
Neither are true, because the basis of both spaces have insufficient amount of linearly independent vectors to match or exceed the dimension of $\mathbb{R}^4$ or $\mathbb{R}^5$, which are 4 and 5 respectively.

[1] weather /ˈwɛðə/ noun - the state of the atmosphere at a particular place and time as regards heat, cloudiness, dryness, sunshine, wind, rain, etc.

[2] `https://www.accuweather.com/en/am/yerevan/16890/weather-forecast/16890`

h) Find the null space null(A), and find a basis for null(A).

$$\begin{cases} x_1 - 3x_2 + x_4 = 0 \\ x_3 + 2x_4 = 0 \\ x_5 = 0 \end{cases} \longrightarrow \begin{cases} x_1 = 3x_2 - x_4 \\ x_3 = -2x_4 \\ x_5 = 0 \end{cases}$$

$$\text{null}(A) = \left\{ \begin{bmatrix} 3s - t \\ s \\ -2t \\ t \\ 0 \end{bmatrix}, s, t \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Exercise 3 (20 points).

Consider the vector space $M_2(\mathbb{R})$, its standard basis $E = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ and the basis $G = \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \right\}$. Let $A = \begin{bmatrix} 0 & 4 \\ 1 & -3 \end{bmatrix} \in M_{22}(\mathbb{R})$

a) Find $[A]_E$ (the coordinate vector of A with respect to E).

$$[A]_E = \begin{bmatrix} 0 \\ 4 \\ 1 \\ -3 \end{bmatrix}$$

b) Find $[A]_G$ (the coordinate vector of A with respect to G).
A is a linear combination of matrices in the set G. A system of linear equations were constructed in the form of an augmented matrix.

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 2 & 4 \\ 1 & 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & -3 \end{array}\right] \longleftrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & -3 \\ 0 & 0 & -2 & 2 & 4 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \longleftrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & -5 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$$\longleftrightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \longleftrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \Longrightarrow [A]_G = \begin{bmatrix} -4 \\ 5 \\ -2 \\ 0 \end{bmatrix}$$

c) Find the change of basis matrix $P_{G \leftarrow E}$ and check the formula $[A]_G = P_{G \leftarrow E}[A]_E$.
Instead of constructing several systems of linear equations of matrices in the set E as linear combinations of matrices in the set G, the coordinate vectors were found by noticing the obvious:

$$\begin{cases} E_1 = 0G_1 + 0G_2 + 1G_3 + 1G_4 \\ E_2 = 0G_1 + 0G_2 + -\frac{1}{2}G_3 + 0G_4 \\ E_3 = -G_1 + 2G_2 + 0G_3 + 0G_4 \\ E_4 = 1G_1 - G_2 + 0G_3 + 0G_4 \end{cases} \Longrightarrow P_{G \leftarrow E} = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & -2 \\ 1 & -\frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$P_{G\leftarrow E}[A]_E = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & 0 & 2 & -1 \\ 1 & -\frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 4 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -4 \\ 5 \\ -2 \\ 0 \end{bmatrix} \overset{\text{conicidentally}}{=} [A]_G$$

Exercise 4 (20 points).

Consider the vector space $P_2(\mathbb{R})$ and its two basis $B = \{1, 1+2x, x-2x^2\}$, $C = \{x^2, 1-3x+x^2, 1-x^2\}$. Let the coordinate vector of the polynomial $p(x) \in P_2(\mathbb{R})$ with respect to B be

$$[p(x)]_B = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$$

a) Find the polynomial $p(x)$.

$$p(x) = (-1)\cdot 1 + 0 \cdot (1+2x) + 3\cdot(x-2x^2) = -1+3x-6x^2$$

b) Find the change-of-basis matrix $P_{C\leftarrow B}$ using the Gauss-Jordan method.

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 0 & -3 & 0 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right] \longleftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \Longrightarrow [1]_C = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 0 & -3 & 0 & 2 \\ 1 & 1 & -1 & 0 \end{array}\right] \longleftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7/3 \\ 0 & 1 & 0 & -2/3 \\ 0 & 0 & 1 & 5/3 \end{array}\right] \Longrightarrow [1+2x]_C = \begin{bmatrix} 7/3 \\ -2/3 \\ 5/3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & -3 & 0 & 1 \\ 1 & 1 & -1 & -2 \end{array}\right] \longleftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4/3 \\ 0 & 1 & 0 & -1/3 \\ 0 & 0 & 1 & 1/3 \end{array}\right] \Longrightarrow [x-x^2]_C = \begin{bmatrix} -4/3 \\ -1/3 \\ 1/3 \end{bmatrix}$$

$$P_{C\leftarrow B} = \begin{bmatrix} [1]_C & [1+2x]_C & [x-2x^2]_C \end{bmatrix} = \begin{bmatrix} 1 & 7/3 & -4/3 \\ 0 & -2/3 & -1/3 \\ 1 & 5/3 & 1/3 \end{bmatrix}$$

c) Find the change-of-basis matrix $P_{B\leftarrow C}$ using the change-of-basis matrix $P_{C\leftarrow B}$.

$$P_{B\leftarrow C} = P_{C\leftarrow B}^{-1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 7/3 & -4/3 & 1 & 0 & 0 \\ 0 & -2/3 & -1/3 & 0 & 1 & 0 \\ 1 & 5/3 & 1/3 & 0 & 0 & 1 \end{array}\right] \longleftrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/4 & 9/4 & 5/4 \\ 0 & 1 & 0 & 1/4 & -5/4 & -1/4 \\ 0 & 0 & 1 & -1/2 & -1/2 & 1/2 \end{array}\right]$$

$$\therefore P_{B\leftarrow C} = \begin{bmatrix} -1/4 & 9/4 & 5/4 \\ 1/4 & -5/4 & -1/4 \\ -1/2 & -1/2 & 1/2 \end{bmatrix}$$

d) Find $[p(x)]_C$ using the appropriate change of basis matrix obtained above.

$$[p(x)]_C = P_{C\leftarrow B} \cdot [p(x)]_B = \begin{bmatrix} 1 & 7/3 & -4/3 \\ 0 & -2/3 & -1/3 \\ 1 & 5/3 & 1/3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ 0 \end{bmatrix}$$

e) Write the change of basis matrices $P_{B\leftarrow E}$ and $P_{C\leftarrow E}$, and check the formula. $P_{C\leftarrow B} = P_{C\leftarrow E}P_{E\leftarrow B}$ ($P_{C\leftarrow B} = P^{-1}_{E\leftarrow C}P_{E\leftarrow B}$).

$$P_{C\leftarrow E} = \begin{bmatrix}[1]_C & [x]_C & [x^2]_C\end{bmatrix} = \begin{bmatrix}1 & 2/3 & 1\\ 0 & -1/3 & 0\\ 1 & 1/3 & 0\end{bmatrix}$$

$$P_{B\leftarrow E} = \begin{bmatrix}[1]_B & [x]_B & [x^2]_B\end{bmatrix} = \begin{bmatrix}1 & -1/2 & -1/4\\ 0 & 1/2 & 1/4\\ 0 & 0 & -1/2\end{bmatrix}$$

$$P_{E\leftarrow B} = \begin{bmatrix}[1]_E & [1+2x]_E & [x-x^2]_E\end{bmatrix} = \begin{bmatrix}1 & 1 & 0\\ 0 & 2 & 1\\ 0 & 0 & -1\end{bmatrix}$$

$$P_{C\leftarrow E}P_{E\leftarrow B} = \begin{bmatrix}1 & 2/3 & 1\\ 0 & -1/3 & 0\\ 1 & 1/3 & 0\end{bmatrix} \cdot \begin{bmatrix}1 & 1 & 0\\ 0 & 2 & 1\\ 0 & 0 & -2\end{bmatrix} = \begin{bmatrix}1 & 7/3 & -4/3\\ 0 & -2/3 & -1/3\\ 1 & 5/3 & 1/3\end{bmatrix} \overset{\text{coincidentally}}{=} P_{C\leftarrow B}$$

Exercise 5 (20 points).

Consider the following real matrices:

$$A = \begin{bmatrix}1 & 0 & 0 & -3\\ 0 & 4 & 2 & 0\\ 1 & 3 & 0 & -2\\ 0 & 0 & 0 & 1\end{bmatrix}, B = \begin{bmatrix}0 & 2 & 1 & 0\\ 1 & 0 & -1 & 1\\ 0 & 0 & 0 & -1\\ 2 & 0 & -1 & 4\end{bmatrix}, C = \begin{bmatrix}3 & 6 & 5 & -1\\ 0 & 1 & -3 & 2\\ 1 & 2 & 1 & -1\\ 2 & 4 & 7 & 2\end{bmatrix}$$

a) Compute $\det A$ by Laplace expansion theorem, using a column, that seems convenient (with the most number of zeros).

$$\det A = -2\begin{vmatrix}1 & 0 & -3\\ 1 & 3 & -2\\ 0 & 0 & 1\end{vmatrix} = -6\begin{bmatrix}1 & -3\\ 0 & 1\end{bmatrix} = -6$$

b) Compute $\det B$ by Laplace expansion theorem, using a row, that seems convenient (with the most number of zeros).

$$\det B = \begin{vmatrix}0 & 2 & 1\\ 1 & 0 & -1\\ 2 & 0 & -1\end{vmatrix} = -2\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix} + 1\begin{bmatrix}1 & 0\\ 2 & 0\end{bmatrix} = -2$$

c) Compute $\det C$ by reducing the triangular form, using row reduction.

$$\begin{bmatrix}3 & 6 & 5 & -1\\ 0 & 1 & -3 & 2\\ 1 & 2 & 1 & -1\\ 2 & 4 & 7 & 2\end{bmatrix} \xleftrightarrow{R_1\leftrightarrow R_3} \begin{bmatrix}1 & 2 & 1 & -1\\ 0 & 1 & -3 & 2\\ 3 & 6 & 5 & -1\\ 2 & 4 & 7 & 2\end{bmatrix} \xleftrightarrow[R_3-3R_1]{R_4-2R_1} \begin{bmatrix}1 & 2 & 1 & -1\\ 0 & 1 & -3 & 2\\ 0 & 0 & 2 & 2\\ 0 & 0 & 5 & 4\end{bmatrix}$$

$$\xleftrightarrow[R_4-5R_2]{\frac{1}{2}R_3} \begin{bmatrix}1 & 2 & 1 & -1\\ 0 & 1 & -3 & 2\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & -1\end{bmatrix}$$

$$\det C = (-1)\cdot 2\cdot(1\cdot 1\cdot 1\cdot -1) = 2$$

Exercise 6 (10 points).

Consider the matrices A, B, C from Exercise 5 above. Use the properties of determinant to complete the points below.

a) Compute the determinant of a matrix A^5B^T and conclude if it is invertible.

$$\det\ A^5B^T = (\det\ A)^5 \cdot \det\ B = 15552 \neq 0 \Longrightarrow A^5B^T \text{ is invertible}$$

b) Compute the determinant of a matrix $(2C)^2A^{-3}$ and conclude if it is invertible.

$$\det\ (2C)^2A^{-3} = 4C^2(A^{-1})^3 = 2^4 \cdot (\det\ C)^2 \cdot \left(\frac{1}{\det\ A}\right)^3 = -\frac{4}{27} \neq 0 \Longrightarrow (2C)^2A^{-3} \text{ is invertible}$$

c) From the points above deduce if $\operatorname{rank}(A^5B^T) = \operatorname{rank}((2C)^2A^{-3})$.
Since they are both invertible, they both have rank of 4, therefore $\operatorname{rank}(A^5B^T) = \operatorname{rank}((2C)^2A^{-3})$.

d) Compute the determinant of the matrix D that can be obtained from C by performing the following sequence of elementary row operations on C:

$$R_1 + 3R_3, R_2 \leftrightarrow R_3, 5R_4, \frac{1}{3}R_2, R_4 - R_1$$

$$\det\ D = (-1) \cdot 5 \cdot \frac{1}{3} \cdot \det\ C = -\frac{10}{3}$$