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CS104 Linear Algebra Section D - Homework 9

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Exercise 1 (20 points).

Use Cramer's Rule to solve the given linear system.

$$\begin{cases} x + 2y - z = 1 \\ x + 2y + z = 2 \\ x - 2y = 3 \end{cases}$$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 1 \\ 1 & -2 & 0 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \det(A) = 1\begin{vmatrix} 2 & 1 \\ -2 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = 8$$

$$A_1(\mathbf{b}) = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 1 \\ 3 & -2 & 0 \end{bmatrix}, \det(A_1(\mathbf{b})) = 1\begin{vmatrix} 2 & 1 \\ -2 & 0 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} - 1\begin{vmatrix} 2 & 2 \\ 3 & -2 \end{vmatrix} = 18$$

$$A_2(\mathbf{b}) = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{bmatrix}, \det(A_2(\mathbf{b})) = 1\begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = -3$$

$$A_3(\mathbf{b}) = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & -2 & 3 \end{bmatrix}, \det(A_3(\mathbf{b})) = 1\begin{vmatrix} 2 & 2 \\ -2 & 3 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} = 4$$

$$\begin{cases} x = \frac{\det(A_1(\mathbf{b}))}{\det(A)} \\ y = \frac{\det(A_2(\mathbf{b}))}{\det(A)} \\ z = \frac{\det(A_3(\mathbf{b}))}{\det(A)} \end{cases} \longrightarrow \begin{cases} x = \frac{18}{8} \\ y = \frac{-3}{8} \\ z = \frac{4}{8} \end{cases} \longrightarrow \begin{cases} x = \frac{9}{4} \\ y = -\frac{3}{8} \\ z = \frac{1}{2} \end{cases}$$

Exercise 2 (20 points).

Use adjoint method to compute the inverse of the coefficient matrix of the exercise 1.

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{8}\begin{bmatrix} \begin{vmatrix} 2 & 1 \\ -2 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ -2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 1 & -2 \end{vmatrix} \\ \begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} \end{bmatrix}^T = \frac{1}{8}\begin{bmatrix} 2 & 1 & -4 \\ 2 & 1 & 4 \\ 4 & -2 & 0 \end{bmatrix}^T = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{8} & \frac{1}{8} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Exercise 3 (?? points).

In each case below verify that T is a linear transformation.

a) $T : \mathbb{R}^3 \to M_{2\times 3}(\mathbb{R})$ is given by $T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & a & -2a \\ b & 0 & a+2b \end{bmatrix}$;

Take $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$, $c \in \mathbb{R}$

$$1)\, T(\mathbf{u}+\mathbf{v}) = T \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix} = \begin{bmatrix} 0 & u_1+v_1 & -2u_1-2v_1 \\ u_2+v_2 & 0 & u_1+v_1+2u_2+2v_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & u_1 & -2u_1 \\ u_2 & 0 & u_1+2u_2 \end{bmatrix} + \begin{bmatrix} 0 & v_1 & --2v_1 \\ v_2 & 0 & v_1+2v_2 \end{bmatrix} = T(\mathbf{u}) + T(\mathbf{v})$$

$$2)\, T(c\cdot\mathbf{u}) = T \begin{bmatrix} c\cdot u_1 \\ c\cdot u_2 \\ c\cdot u_3 \end{bmatrix} = \begin{bmatrix} 0 & c\cdot u_1 & -2c\cdot u_2 \\ c\cdot u_2 & 0 & c\cdot u_1 + 2c\cdot u_2 \end{bmatrix} = c \begin{bmatrix} 0 & u_1 & -2u_2 \\ u_2 & 0 & u_1+2u_2 \end{bmatrix} = cT(\mathbf{u})$$

$\therefore T$ is a linear transformation.

b) $T : M_2(\mathbb{R}) \to P_3(\mathbb{R})$ is given by $T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (2a+b)x - (b-2c)x^3$;

Take $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \in M_2(\mathbb{R}), c \in \mathbb{R}$

$$1)\, T(A+B) = T \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix} = (2(a_{11}+b_{11}) + a_{12}+b_{12})x - (a_{12}+b_{12} - 2(a_{21}+b_{21}))x^3$$

$$(2a_{11}+a_{12})x - (a_{12}-2a_{21})x^3 + (2b_{11}+b_{12})x - (b_{12}-2b_{21})x^3 = T(A) + T(B)$$

$$2)\, T(cA) = T \begin{bmatrix} c\cdot a_{11} & c\cdot a_{12} \\ c\cdot a_{21} & c\cdot a_{22} \end{bmatrix} = (2c\cdot a_{11} + c\cdot a_{12})x - (c\cdot a_{12} - 2c\cdot a_{21})x^3 = cT(A)$$

$\therefore T$ is a linear transformation.

c) $T : P_2(\mathbb{R}) \to \mathbb{R}^2$ is given by $T(p(x)) = \begin{bmatrix} 3p(1) \\ p(0) \end{bmatrix}$.

Take $a(x) = a_0 + a_1x + a_2x^2, b(x) = b_0 + b_1x + b_2x^2 \in P_2(\mathbb{R}), c \in \mathbb{R}$

$$1)\, T(a(x)+b(x)) = T((a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2) = \begin{bmatrix} 3((a_0+b_0)+(a_1+b_1)+(a_2+b_2)) \\ a_0+b_0 \end{bmatrix}$$

$$= \begin{bmatrix} 3(a_0+a_1+a_2) \\ a_0 \end{bmatrix} + \begin{bmatrix} 3(b_0+b_1+b_2) \\ b_0 \end{bmatrix} = T(a(x)) + T(b(x))$$

$$2)\, T(c\cdot a(x)) = \begin{bmatrix} 3(c\cdot a_0 + c\cdot a_1 + c\cdot a_2) \\ c\cdot a_0 \end{bmatrix} = cT(a(x))$$

$\therefore T$ is a linear transformation.

Exercise 4 (?? points).

Show that the given mapping is a nonlinear transformation.

a) $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ defined by $T(A) = A^3$;
Take $A \in M_2(\mathbb{R}), c \in \mathbb{R}$

$$T(cA) = (cA)^3 = c^3 A^3 \neq cA^3 = cT(A)$$

$$\therefore T \text{ is not a linear transformation.}$$

b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x_1, x_2) = (x_1, x_1 + x_2 + 1)$.
Take $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2, c \in \mathbb{R}$

$$T(c\mathbf{u}) = (c \cdot u_1, c \cdot u_1 + c \cdot u_2 + 1) \neq (c \cdot u_1, c \cdot u_1 + c \cdot u_2 + c) = cT(\mathbf{u})$$

$$\therefore T \text{ is not a linear transformation.}$$

Exercise 5 (?? points).

Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be a linear transformation, satisfying

$$T(1) = 1 - x,\ T(1 + x) = x + x^2,\ T(x + x^2) = 1 - 2x + 3x^2.$$

For the vector $p(x) = 5 + 3x - x^2$ in $P_2(\mathbb{R})$, determine $T(p(x))$.

$$T(x) = T(1 + x) - T(1) = x + x^2 - 1 + x = -1 + 2x + x^2$$

$$T(x^2) = T(x + x^2) - T(x) = 1 - 2x + 3x^2 + 1 - 2x - x^2 = 2 - 4x + 2x^2$$

$$T(5 + 3x - x^2) = 5T(1) + 3T(x) - T(x^2) = 5 - 5x - 3 + 6x + 3x^2 - 2 + 4x - 2x^2 = 5x + x^2$$

Exercise 6 (?? points).

Determine the standard matrix of the given transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 + 3x_3 \\ x_3 - 2x_1 \end{bmatrix}$$

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix},\ T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix},\ T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 0 & 1 \end{bmatrix}$$

Exercise 7 (?? points).

Consider the linear transformations in **Exercise 3 (a-c)** and find their standard matrices.

a) $T : \mathbb{R}^3 \to M_{2\times 3}(\mathbb{R})$ is given by $T\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & a & -2a \\ b & 0 & a+2b \end{bmatrix}$;

The coordinate vectors of both vector spaces were used with their corresponding standard basis.

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}, T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

b) $T : M_2(\mathbb{R}) \to P_3(\mathbb{R})$ is given by $T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (2a+b)x - (b-2c)x^3$;

The coordinate vectors of both vector spaces were used with their corresponding standard basis.

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, T\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, T\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}, T\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{bmatrix}$$

c) $T : P_2(\mathbb{R}) \to \mathbb{R}^2$ is given by $T(p(x)) = \begin{bmatrix} 3p(1) \\ p(0) \end{bmatrix}$.

$$T(1) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, T(x) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, T(x^2) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 3 & 3 & 3 \\ 1 & 0 & 0 \end{bmatrix}$$

Exercise 8 (?? points).

Assume that T defines a linear transformation $T : \mathbb{R}^2 \to P_2(\mathbb{R})$, such that

$$T\begin{bmatrix} 1 \\ -3 \end{bmatrix} = 2 - 3x - x^2, \text{ and } T\begin{bmatrix} 4 \\ 0 \end{bmatrix} = 4x - 8x^2$$

a) Find $T\begin{bmatrix}1\\0\end{bmatrix}$ and $T\begin{bmatrix}0\\-3\end{bmatrix}$;

$$T\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{4}T\begin{bmatrix}4\\0\end{bmatrix} = x - 2x^2$$

$$T\begin{bmatrix}0\\-3\end{bmatrix} = T\begin{bmatrix}1\\-3\end{bmatrix} - T\begin{bmatrix}1\\0\end{bmatrix} = 2 - 4x + x^2$$

b) Find $T\begin{bmatrix}0\\1\end{bmatrix}$, and hence the standard matrix $[T]$ of T;

$$T\begin{bmatrix}0\\1\end{bmatrix} = -\frac{1}{3}T\begin{bmatrix}0\\-3\end{bmatrix} = -\frac{2}{3} + \frac{4}{3}x - \frac{1}{3}x^2$$

Using the standard basis for $P_2(\mathbb{R})$, we get:

$$[T] = \begin{bmatrix}0 & -\frac{2}{3}\\ 1 & \frac{4}{3}\\ -2 & -\frac{1}{3}\end{bmatrix}$$

c) Compute $T\begin{bmatrix}2\\-5\end{bmatrix}$ using the standard matrix of T obtained in point **b)**;

$$T\begin{bmatrix}2\\-5\end{bmatrix} = \begin{bmatrix}0 & -\frac{2}{3}\\ 1 & \frac{4}{3}\\ -2 & -\frac{1}{3}\end{bmatrix}\begin{bmatrix}2\\-5\end{bmatrix} = \begin{bmatrix}\frac{10}{3}\\ -\frac{14}{3}\\ -\frac{7}{3}\end{bmatrix}$$

Exercise 1 (0 points).

Prove that there is no linear transformation $T : \mathbb{R}^2 \to P_2(\mathbb{R})$, such that

$$T\begin{bmatrix}1\\-2\end{bmatrix} = 1 - x^2,\ T\begin{bmatrix}-2\\5\end{bmatrix} = x + 4x^2 \text{ and } T\begin{bmatrix}2\\-3\end{bmatrix} = 4 + x + x^2$$

Since $\dim(\mathbb{R}^2) = 2$, $\exists c_1, c_2 \in \mathbb{R}$, such that $c_1\begin{bmatrix}1\\-2\end{bmatrix} + c_2\begin{bmatrix}-2\\5\end{bmatrix} = \begin{bmatrix}2\\-3\end{bmatrix}$

$$\left[\begin{array}{cc|c}1 & -2 & 2\\ -2 & 5 & -3\end{array}\right] \longleftrightarrow \left[\begin{array}{cc|c}1 & -2 & 2\\ 0 & 1 & 1\end{array}\right] \longleftrightarrow \left[\begin{array}{cc|c}1 & 0 & 4\\ 0 & 1 & 1\end{array}\right] \implies \begin{bmatrix}2\\-3\end{bmatrix} = 4\begin{bmatrix}1\\-2\end{bmatrix} + \begin{bmatrix}-2\\5\end{bmatrix}$$

$$T\begin{bmatrix}2\\-3\end{bmatrix} = 4T\begin{bmatrix}1\\-2\end{bmatrix} + T\begin{bmatrix}-2\\5\end{bmatrix} = 4 - 4x^2 + x + 4x^2 = 4 + x \neq 4 + x + x^2$$

$$\therefore T \text{ is not a linear transformation.}$$