Digital version

CS105 Ordinary Differential Equations Section B - Homework 2

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Problem 1.

Solve the following linear 1st order ODEs:

a) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 6t - \dfrac{6y}{t}$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 6t - \frac{6y}{t} \implies \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{6y}{t} = 6t \implies \mu(t)\frac{\mathrm{d}y}{\mathrm{d}t} + \mu(t)\frac{6y}{t} = \mu(t)6t$$

$$(\mu(t)y)' = \mu(t)\frac{\mathrm{d}y}{\mathrm{d}t} + \mu'(t)y \implies \mu'(t) = \mu(t)\frac{6}{t} \implies \frac{\mu'(t)}{\mu(t)} = \frac{6}{t} \implies \ln(\mu(t)) = 6\ln(t) \implies \mu(t) = t^6$$

$$(t^6y)' = 6t^7 \implies y = \frac{\frac{6t^8}{8} + c}{t^6} \therefore y(t) = \frac{3t^2}{4} + \frac{c}{t^6}$$

b) $\dfrac{\mathrm{d}y}{\mathrm{d}t} - 2ty = t$ with $y(0) = 1/2$

$$\mu(t)\frac{\mathrm{d}y}{\mathrm{d}t} - 2\mu(t)ty = \mu(t)t \implies (\mu(t)y)' = \mu'(t)\frac{\mathrm{d}y}{\mathrm{d}t} + \mu'(t)y$$

$$\mu'(t) = -2\mu(t)t \implies \frac{\mu'(t)}{\mu(t)} = -2t \implies \mu(t) = e^{-t^2} \implies (e^{-t^2}y)' = te^{-t^2}$$

$$e^{-t^2}y = -\frac{1}{2}\int e^{-t^2}\,\mathrm{d}(-t^2) = \frac{-\frac{1}{2}e^{-t^2} + c}{e^{-t^2}} \therefore y(t) = ce^{t^2} - \frac{1}{2}$$

$$y(0) = 0.5 \implies 1 = ce^0 \implies c = 1 \implies y = e^{t^2} - \frac{1}{2}$$

Problem 2.

Solve the following separable ODEs:

a) $\dfrac{\mathrm{d}y}{\mathrm{d}z} = (1+z)(1+y^2)$ with $y(0) = 0$

$$\frac{y'}{1+y^2} = 1+z \implies \int \frac{\mathrm{d}y}{1+y^2} = \int 1+z\,\mathrm{d}z \implies \arctan(y) = z + \frac{z^2}{2} + c \implies y = \tan\left(\frac{2z+z^2}{2} + c\right)$$

$$y(0) = \tan(c) = 0 \implies c = \pi k, k \in \mathbb{N}$$

b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = e^{4y-x}$ with $y(4) = 2 - \ln 2$

$$\frac{y'}{e^{4y}} = e^{-x} \implies \int \frac{\mathrm{d}y}{e^{4y}} = \int \frac{\mathrm{d}x}{e^{x}} \implies \frac{1}{4e^{4y}} = \frac{1}{e^{x}} + c \implies e^{4y} = \frac{e^{x}}{4 + 4ce^{x}}$$

$$y = \frac{x}{4} - \frac{1}{4}\ln\left(4 + 4ce^{x}\right), y(4) = 2 - \ln 2 \implies c = \frac{2}{e^{5}} = \frac{4}{e^{4}}$$

Problem 3.

a) Solve $\dfrac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 1$. You should obtain $y(x) = \dfrac{1 + Ce^{2x}}{1 - Ce^{2x}}$

$$\frac{y'}{y^2 - 1} = 1 \implies \int \frac{\mathrm{d}y}{(y-1)(y+1)} = x + c_1 \implies \ln\left(\frac{y-1}{y+1}\right) = 2x + c$$

$$y - 1 = ye^{2x+c} + e^{2x+c} \implies y(x) = \frac{1 + Ce^{2x}}{1 - Ce^{2x}}$$

b) What are the equilibrium solutions? (hint, look at the ODE). Notice that one of the equilibrium solutions does not belong to the general solution you found above.

$$c = 0 \implies y(x) = 1, c \neq 0 \implies \lim_{x \to -\infty} y(x) = 1, \lim_{x \to \infty} y(x) = -1$$

c) Notice that each solution with has $C > 0$ a vertical asymptote. For $x = 0$ identify the range of initial conditions that result in solutions that "blow up" (have a pole/singularity) at finite value of x.

$$y(0) = \frac{1 + C}{1 - C} \implies C \in (0, +\infty) \therefore y(0) \in (-\infty, -1) \cup (1, \infty)$$

Problem 4.

Solve the following ODEs using substitution techniques:

a) $z' + xz = xz^4$

$$z' + xz = xz^4 \implies \frac{z'}{z^4} + \frac{x}{z^3} = x$$

$$u = z^{-3} \implies u' = -3\frac{z'}{z^4} \implies -\frac{u'}{3} + ux = x \implies u' - 3ux = -3x$$

$$\mu(x)u' - 3\mu(x)ux = -3\mu(x)x \implies (\mu(x)u)' = \mu(x)u' + \mu'(x)u \implies \mu'(x)u = -3\mu(x)ux$$

$$\frac{\mu'(x)}{\mu(x)} = -3x \implies \mu(x) = e^{-1.5x^2} \implies u = \frac{1}{e^{-1.5x^2}} \int e^{-1.5x^2}\, \mathrm{d}(-1.5x^2) = 1 + ce^{1.5x^2}$$

$$z = u^3 \therefore z = (1 + ce^{1.5x^2})^3$$

b) $t^2 + 3ty + y^2 - t^2\dfrac{\mathrm{d}y}{\mathrm{d}t} = 0$

$$y = tu \implies y' = u't + u \implies t^2 + 3t^2u + t^2u^2 - t^2(u't + u) = 0 \implies u't - 4u - u^2 + 1 = 0$$

$$u' = \frac{u^2 + 4u - 1}{t} \implies \frac{u'}{u^2 + 4u - 1} = \frac{1}{t} \implies \ln\left|\frac{u + \frac{4-2\sqrt{5}}{2}}{u - \frac{4-2\sqrt{5}}{2}}\right| = 2\sqrt{5}\ln(t) + c_1$$

$$\frac{u + \frac{4-2\sqrt{5}}{2}}{u - \frac{4-2\sqrt{5}}{2}} = c_2 t^{2\sqrt{5}} \implies u = \frac{-\frac{4-2\sqrt{5}}{2} - c_2 t^{2\sqrt{5}}}{1 - c_2 t^{2\sqrt{5}}} \implies y = t\frac{-\frac{4-2\sqrt{5}}{2} - c_2 t^{2\sqrt{5}}}{1 - c_2 t^{2\sqrt{5}}}$$