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CS105 Ordinary Differential Equations Section B - Homework 3

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Problem 1

Find a curve in xy-plane that passes through the point $(0, 3)$ and whose tangent line at the point (x, y) has slope of $2x/y^2$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{y^2} \overset{y'\,\mathrm{d}x=\mathrm{d}y}{\Longrightarrow} \int y^2 \,\mathrm{d}y = \int 2x \,\mathrm{d}x \implies y^3 = 3x^2 + c \implies y = \sqrt[3]{3x^2 + c}$$

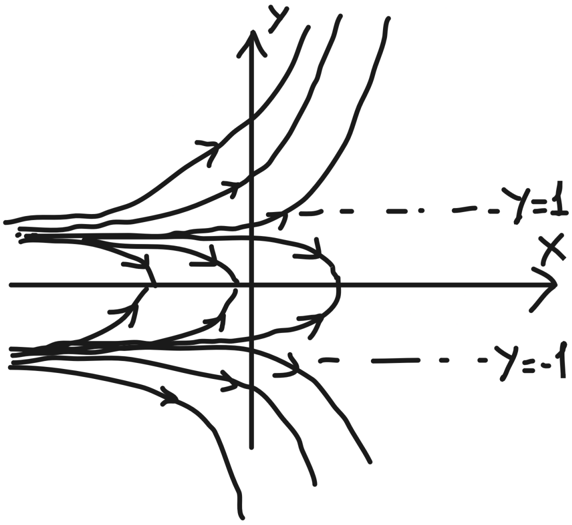
$$x = 0, y = 3 \implies 3 = \sqrt[3]{0 + c} \implies c = 3^3 \therefore y = \sqrt[3]{3x^3 + 27}$$

Problem 2

Characterize the family of solution of the ODE $y' = \dfrac{y^2 - 1}{2y}$ that can be obtained with integration. Note that this set of solutions does not include the equilibrium solutions. Find those as well. Draw a representative diagram of the integral curves.

$$y' = \frac{y^2 - 1}{2y} \implies \int \frac{2y}{y^2 - 1} \,\mathrm{d}y = \int \mathrm{d}t \implies \ln \left|y^2 - 1\right| = t + c \implies y^2 = 1 + Ce^t, y \notin \{-1, 1\}$$

This gives two family of solutions, $y = \sqrt{1 + Ce^t}$ and $y = -\sqrt{1 + Ce^t}$. Equilibrium is $y = \pm 1$.



Problem 3

The time of death of a murdered person can be estimated using an ODE-based model. The body is discovered in a room maintained at a constant temperature of $21°C$. According to Newton's Law of Cooling, the rate at which the body loses heat is proportional to thetemperature difference between the body and the surrounding environment. Let $T(t)$ represent the body temperature at time t (in hours). Then, for some proportionality constant k, the temperature evolution is described by the differential equation:

$$T'(t) = k(21 - T(t))$$

When the police arrive at the crime scene, they measure the body temperature at $24°C$. Their goal is to determine how many hours have passed since the time of death. In forensic science, the cooling constant k is typically assumed to fall within the range $[0.4, 0.6]$ per hour. Given that a living person's normal body temperature is $37°C$, determine the possible range of estimates for the time elapsed since death.

$$T'(t) = k(21 - T(t)) \implies T'(t) + kT(t) = 21k \implies \mu(t)T'(t) + \mu(t)kT(t) = 21\mu(t)k$$

$$(\mu(t)T(t))' = \mu(t)T'(t) + \mu'(t)T(t) \implies \mu'(t) = k\mu(t) \implies \mu(t) = e^{kt}$$

$$(e^{kt}T(t))' = 21ke^{kt} \implies e^{kt}T(t) = 21k^2e^{kt} + c \implies T(t) = 21k^2 + Ce^{-kt}$$

$$k_1 = 0.4, T(0) = 37 \implies 37 = 21 \cdot 0.4^2 + C \implies C = 33.64 \implies T(t) = 3.36 + 33.64e^{-0.4t}$$

$$k_2 = 0.6, T(0) = 37 \implies 37 = 21 \cdot 0.6^2 + C \implies C = 33.64 \implies T(t) = 7.56 + 29.44e^{-0.6t}$$

$$T(t) = 24 \implies t_1 \approx 1.22 \text{ hours}, t_2 \approx 0.97 \text{ hours}$$

Problem 4

The rate at which an epidemic spreads through a community is jointly proportional to the number of residents who have been infected and the number of susceptible residents who have not. Express the number of residents who have been infected as a function of time. Assume the number of residents is B.

Denote the number of infected $i(t)$ and the proportionality constant k.

$$i' = ki(B - i) \implies \frac{i'}{ki(B - i)} = 1 \implies \int 1 \, \mathrm{d}t = \frac{1}{k} \int \frac{\mathrm{d}i}{i(B - i)}$$

$$\frac{1}{i(b - i)} = \frac{a}{i} + \frac{b}{B - i} \implies aB - ai + bi \equiv 1 \implies a = b = \frac{1}{B}$$

$$t = \frac{1}{kB} \ln \left| \frac{i}{B - i} \right| + c_1 \implies c_2e^{tkB} = \frac{i}{B - i} \implies i = \frac{Bc_2e^{tkB}}{1 + c_2e^{tkB}}$$

$$\therefore i(t) = \frac{B}{1 + Ce^{-tkB}}$$