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CS105 Ordinary Differential Equations Section B - Homework 4

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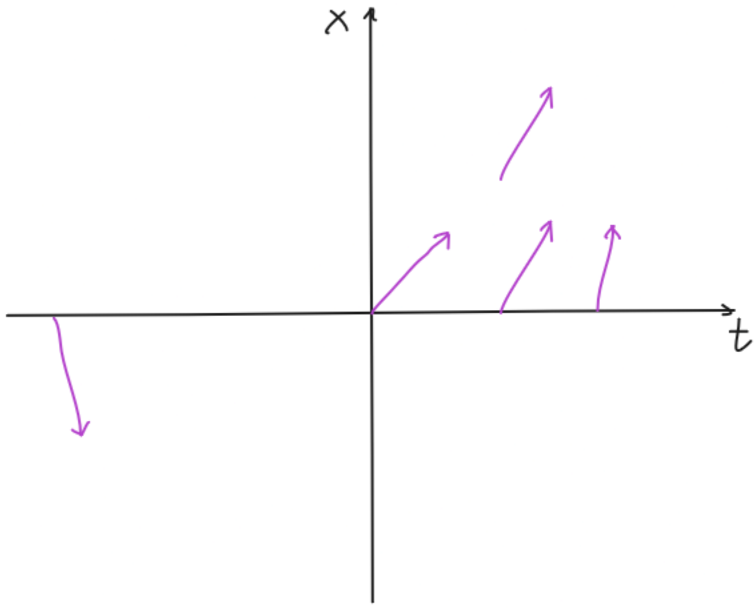
February 24, 2025

Problem 1: Playing with Euler's Method

a) Draw the direction field vectors of $\dfrac{\mathrm{d}x}{\mathrm{d}t} = t^3 + 1$ at the points $(t, x) = (0, 0)$, $(1, 0)$, $(1, 1)$, $(-2, 0)$ and (1.5,0) by hand. Give the angles of the vectors with respect to the t-axis in radians or degrees (round to 3 significant figures).



The angles are as follows: $\dfrac{\pi}{4}$, $\arctan(2) \approx 1.11$, 1.11, $\arctan(-7) \approx -1.43$,$\arctan\left(1.5^3 + 1\right) \approx 1.35$.

b) Explain why the horizontal components of the vectors of the direction field cannot point in the negative/left direction.

Since we draw the vectors while increasing the value of t (going from left to right), the Δt is always positive, which is why the horizontal components of the vectors cannot point left. We draw them as vectors, but actually we only use the slopes to visualize the possible solutions.

c) Use a single step of the Euler's method to estimate the value of the solution at a point $t_0 + h$ for each of the points in (a) above. Here t_0 is the value of t that you start with – so for the point $(1, 1)$ you have $t_0 = 1$. Use different values of $h : h = 1, 0.1, 0.001, 0.0001$

	A	B	C	D	E	F
1	t	x	h=1	h=0.1	h=0.001	H=0.0001
2	0	0	1	0.1	0.001	0.0001
3	1	0	2	0.2	0.002	0.0002
4	1	1	3	1.2	1.002	1.0002
5	-2	0	-7	-0.7	-0.007	-0.0007
6	1.5	0	4.375	0.4375	0.004375	0.0004375

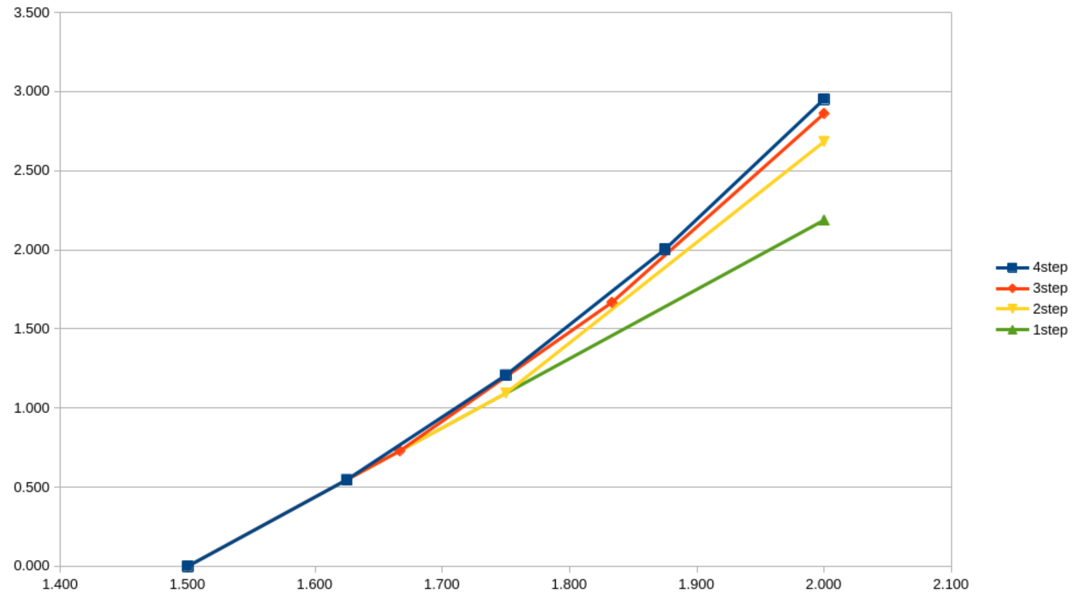
d) Compute the error (deviation from the true value of the solution) for different values of h above. Observe how the error increases as h increases.

$$x' = t^3 + 1 \implies x(t) = \frac{1}{4}t^4 + t + c$$

	A	B	C	D	E	F	G
1	t	x	h=1	h=0.1	h=0.001	H=0.0001	
2	0	0	1.0000	0.1000	0.0010	0.0001	estimate
3	0	0	1.2500	0.1000	0.0010	0.0001	Solution: c=0
4	1	0	2.0000	0.2000	0.0020	0.0002	estimate
5	1	0	4.7500	0.2160	0.0020	0.0002	Solution: c=-5/4
6	1	1	3.0000	1.2000	1.0020	1.0002	estimate
7	1	1	5.7500	1.2160	1.0020	1.0002	Solution: c=-1/4
8	-2	0	-7.0000	-0.7000	-0.0070	-0.0007	estimate
9	-2	0	-2.7500	-0.6420	-0.0070	-0.0007	Solution: c=-2
10	1.5	0	4.3750	0.4375	0.0044	0.0004	estimate
11	1.5	0	9.5000	0.4728	0.0044	0.0004	Solution: c=-177/64

e) Starting at the point $(1.5, 0)$ use Euler's method to estimate the value of the solution at $t = 2$. Use $n = 1, 2, 3, 4$ steps. You need to compute the value of h for each number of steps to end up at the correct final t-coordinate. Namely, if you are using $n = 2$, then $h = \frac{0.5}{2} = 0.25$, so you end up at $t = 2$ after two steps. Plot the 4 different curves next to the solution. You can use Excel for this. Observe how using a greater number of steps improves the accuracy.

	A	B	C	D	E	F	G	H	I
1	n	1		2		3		4	
2	h	0.500		0.250		0.167		0.125	
3		t	x	t	x	t	x	t	
4	0	1.500	0.000	1.500	0.000	1.500	0.000	1.500	0.000
5	1	2.000	2.188	1.750	1.094	1.667	0.729	1.625	0.547
6	2			2.000	2.684	1.833	1.667	1.750	1.208
7	3					2.000	2.861	1.875	2.003
8	4							2.000	2.952



f) I would like you to solve this problem by writing a code. You cannot escape programming tasks if you want to do engineering. Do problem 15 on page 103 from the main textbook.

```
import numpy as np
import matplotlib.pyplot as plt

deltaT = 0.05
N = 22

t0 = 1
x0 = 0

def f(t,x):
    return 3*(t**2)/(3*(x**2)-4)

def euler_method(t0,x0, N):
    arr_t = np.zeros(N)
    arr_x = np.zeros(N)

    arr_t[0] = t0
    arr_x[0] = x0

    for i in range(N-1):
        arr_t[i+1] = arr_t[i] + deltaT
        arr_x[i+1] = arr_x[i] + f(arr_t[i],arr_x[i])*deltaT

    print(arr_t,arr_x)
    return (arr_t,arr_x)

euler_method(t0,x0,N)
```

a) $-0.16613447, -0.41087231, -0.80466017, 4.15867095$

b) $-0.17465201, -0.43423846, -0.88913973, -3.09810486$

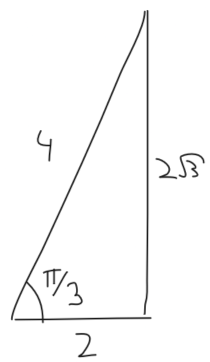
c) For the first three points, the estimations are close, however for the $t = 1.8$ case, the value of y is wildly different. The derivative function probably crossed the value of 0 and we have change of curvature.

Problem 2: Playing with Complex Numbers

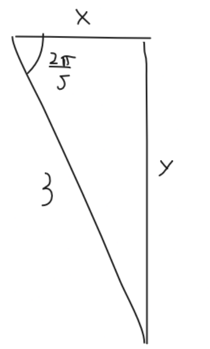
In parts (a) and (b) draw the right triangle with the sides x, y where $z = x+iy$ and label all sides (including hypotenuse) as well as the angle from the polar form with the corresponding value.

a) Convert from Polar to Cartesian form: $4e^{i\frac{\pi}{3}}$, $\dfrac{3}{e^{i\frac{2\pi}{5}}}$

$$4e^{i\frac{\pi}{3}} = 4(\cos(\pi/3) + i\sin(\pi/3)) = 2 + 2\sqrt{3}i$$

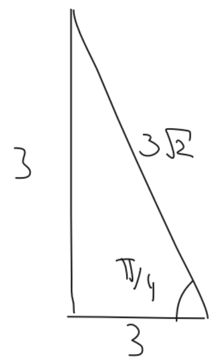


$$\frac{3}{e^{i\frac{2\pi}{5}}} = 3(\cos(-2pi/5) + i\sin(-2pi/5)) = 3\left(\cos\left(\frac{2\pi}{5}\right) - i\sin\left(\frac{2\pi}{5}\right)\right)$$

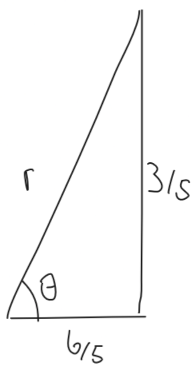


b) Convert from Cartesian to Polar form: $-3+3i$, $\dfrac{3}{2-i}$

$$-3+3i = r\sqrt{2}e^{\frac{3\pi}{4}i}$$



$$r = \frac{\sqrt{3^2+6^2}}{5} = \frac{3\sqrt{5}}{5}, \tan\theta = \frac{3}{6} = \frac{1}{2} \implies \theta = \arctan 0.5 \implies z = \frac{3\sqrt{5}}{5} e^{\arctan(0.5)i}$$



c) Evaluate/simplify using Polar form: $(1+i)^{10}$

$$1+i = \sqrt{2}e^{\frac{\pi}{4}i} \implies (1+i)^{10} = (\sqrt{2}e^{\frac{\pi}{4}i})^{10} = 2^5 e^{\frac{10\pi}{4}i} = 2^5 e^{\frac{\pi}{2}i} = 32i$$

d) Find $Re[(1+i)^{10}]$ and $Im[(1+i)^{10}]$

The real part of the complex number is 0, and the "imaginary" part is 32.

e) Prove the trigonometric identity $\cos(A+B) = \cos A \cos B - \sin A \sin B$ using the trick of taking the real part of a complex identity.

$$e^{i(A+B)} = e^{iA}e^{iB} \implies \cos(A+B) + i\sin(A+B) = (\cos A + i\sin A)(\cos B + i\sin B)$$

$$\cos(A+B) + i\sin(A+B) = \cos A\cos B + i\cos A\sin B + i\sin A\cos B - \sin A\sin B$$

We we consider the real and imaginary parts separately, we get:

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \text{ and } \sin(A+B) = \cos A\sin B + \sin A\cos B$$

f) Consider an inductor-resistor circuit in a series with a battery. The differential equation describing the evolution of the current is given by $L\dfrac{\mathrm{d}I}{\mathrm{d}t} + RI = V(t)$. If the battery is turned on at $t = 0$ and applies the voltage $V(t) = 2\sin\left(\frac{\pi}{2}t + \frac{\pi}{4}\right)$, give the steady-state response of the current. Use the complex form of the input to obtain the complex solution, and then take the real part, as we did in class.

$$2\sin\left(\frac{\pi}{2}t + \frac{\pi}{4}\right) = Im\left[2e^{i\left(\frac{\pi}{2}t+\frac{\pi}{4}\right)}\right], \text{Take } I(t) = I_0 e^{i\left(\frac{\pi}{2}t+\frac{\pi}{4}\right)}$$

$$\left(Li\left(\frac{\pi}{2}\right) + R\right) I_0 e^{i\left(\frac{\pi}{2}t+\frac{\pi}{4}\right)} = 2e^{i\left(\frac{\pi}{2}t+\frac{\pi}{4}\right)} \implies I_0 = \frac{2}{Li\left(\frac{\pi}{2}\right) + R}$$

$$\therefore I(t) = \frac{2}{\frac{\pi}{2}Li + R} e^{i\left(\frac{\pi}{2}t+\frac{\pi}{4}\right)}, \frac{2}{\frac{\pi}{2}Li + R} = \frac{2}{\sqrt{\left(\frac{L\pi}{2}\right)^2 + R^2}} e^{i\arctan\left(\frac{L\pi}{2R}\right)}$$

$$\therefore Re(I(t)) = \frac{2}{\sqrt{\left(\frac{L\pi}{2}\right)^2 + R^2}} \cos\left(\frac{\pi}{2}t + \frac{\pi}{4} + \arctan\left(\frac{L\pi}{2R}\right)\right)$$

Problem 3: Playing with Domains

a) Solve the ODE $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{2}{\sqrt{y}}$ with $y(0) = 9$. What is the range of x for which the solution is valid? Hint: look at the sign of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

$$y \neq 0, \sqrt{y}\,\mathrm{d}y = 2\,\mathrm{d}x \implies \frac{2}{3}y^{\frac{3}{2}} = 2x + C \implies y^{\frac{3}{2}} = 3x + c, y(0) = 9 \overset{3x+c>0}{\implies} y = (3x+27)^{\frac{2}{3}}$$

$$y > 0 \implies 3x + 27 > 0 \implies x \in (-9, +\infty)$$

b) Optional (5% extra credit if the argument is convincing). The ODE $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \Theta(x)$, where $\Theta(x)$ is the Heaviside step function, is undefined at $x = 0$. Argue why this is the case.

$$\Theta(x) = \begin{cases} 1, x \geq 0 \\ 0, x < 0 \end{cases}, \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=0} = \lim_{h\to 0} \frac{\Theta(h) - \Theta(0)}{h}$$

If we approach from the positive x direction, $\Theta(h) \to 1$, which implies that $\Theta(h) - \Theta(0) = 0$, therefore $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$. However, if we approach from the $-x$ direction, we get $\Theta(h) \to 0$, which implies that $\Theta(h) - \Theta(0) = -1$, which means that $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is undefined at $x = 0$. Therefore the ODE is undefined at $x = 0$.