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CS105 Ordinary Differential Equations Section B - Homework 5

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Part 1: Linear Independence and the Wronskian

$$3 : e^{\lambda t}\cos\mu t = Ae^{\lambda t}\sin\mu t \implies \cos\mu t = A\sin\mu t \implies \text{ linearly independent}$$

$$4 : e^{3x-1} = \frac{1}{3}e^{3x} \implies \text{ linearly dependent}$$

12. If the functions y_1 and y_2 are linearly independent solutions of $y'' + p(t)y' + q(t)y = 0$, prove that $y_3 = y_1 + y_2$ and $y_4 = y_1 - y_2$ also form a linearly independent set of solutions. Conversely, if y_3 and y_4 are linearly independent solutions of the differential equation, show that y_1 and y_2 are also.

Since all the solutions can be clasified as $y = c_1y_1 + c_2y_2$, for y_3 take $c_1 = 1, c_2 = 1$ and for y_4 take $c_1 = 1, c_2 = -1$. y_3 and y_4 are linearly independent, because we cannot get the other by multiplying one with a constant.

$$\begin{cases} y_3 = y_1 + y_2 \\ y_4 = y_1 - y_2 \end{cases} \implies \begin{cases} y_1 = \frac{y_3+y_4}{2} \\ y_2 = \frac{y_3-y_4}{2} \end{cases} \implies \text{ linearly independent}$$

27. Show that t and t^2 are linearly independent on $-1 < t < 1$; indeed, they are linearly independent on every interval. Show also that $W(t, t^2)$ is zero at $t = 0$. What can you conclude from this about the possibility that t and t^2 are solutions of a differential equation $y'' + p(t)y' + q(t)y = 0$? Verify that t and t^2 are solutions of the equation $t^2y'' - 2ty' + 2y = 0$. Does this contradict your conclusion? Does the behavior of the Wronskian of t and t^2 contradict Theorem 3.3.2?

$$W(t, t^2) = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = 2t^2 - t^2 = t^2 \implies \text{ independent}, t = 0 \implies W(t, t^2) = 0$$

Therefore t^2 and t cannot be both a solution of the same ODE in the form shown above.

$$2t^2 - 4t^2 + 2t^2 = 0 \text{ and } -2t + 2t = 0 \implies \text{ both satisfy the ODE}$$

Does not contradict the Theorem, as we do not have a linear second order homogeneous ODE.

Part 2: Solving second order linear homogeneous ODEs with constant coefficients

17: Find the differential equation whose general solution is $y = c_1e^{2t} + c_2e^{-3t}$.

Assume the differential equation is $Ay'' + By' + Cy = 0$. Plug in the solution:

$$A(4c_1e^{2t} + 9c_2e^{3t}) + B(2c_1e^{2t} - 3c_2e^{-3t}) + C(c_1e^{2t} + c_2e^{-3t}) = 0$$

$$\text{Assume } A = 1 \implies \begin{cases} 2B + C = -4 \\ -3B + C = -9 \end{cases} \implies \begin{cases} A = 1 \\ B = 1 \\ C = -9 \end{cases} \implies y'' + y - 6y = 0$$

22: $y'' + 2y' + 2y = 0, y(\pi/4) = 2, y'(\pi/4) = -2$

$$\text{Assume } y = e^{rx} \implies (r^2 + 2r + 2)e^{rt} = 0 \implies r = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 2}}{2} = -1 \pm i$$

$$y = a_1e^{(-1+i)t} + a_2e^{(-1-i)t} = e^{-t}(a_1e^{it} + a_2e^{-it}) = e^{-t}((a_1 + a_2)\cos(t) + (a_1 - a_2)i\sin(t))$$

$$y = e^{-t}(c_1\cos(t) + c_2\sin(t)) \implies y(\pi/4) = e^{-\pi/4}(c_1\frac{\sqrt{2}}{2} + c_2\frac{\sqrt{2}}{2}) = 2$$

$$y' = e^{-t}((c_2 - c_1)\cos(t) - (c_1 + c_2)\sin(t)) \implies y'(\pi/4) = e^{-\pi/4}((c_2 - c_1)\frac{\sqrt{2}}{2} - (c_1 + c_2)\frac{\sqrt{2}}{2}) = -2$$

$$\begin{cases} c_1 + c_2 = 2\sqrt{2}e^{\pi/4} \\ -2c_1 = -2\sqrt{2}e^{\pi/4} \end{cases} \implies \begin{cases} c_1 = \sqrt{2}e^{\pi/4} \\ c_2 = \sqrt{2}e^{\pi/4} \end{cases}$$

$$\therefore y(t) = \sqrt{2}e^{-t+\pi/4}\cos(t) + \sqrt{2}e^{-t+\pi/4}\sin(t)$$

Part 3: Analyzing a Circuit

The dynamics of the charge on the capacitor in an R-L-C circuit connected to a battery is given by the following differential equation:

$$L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{1}{C}Q = V(t)$$

At time $t = 0$ the voltage input is set to zero: $V(t) = 0$ for $t \geq 0$. In practice this can be done by using a bypass wire to the battery with a switch on it, initially in an open mode (so no current passes through the bypass wire). As the battery is turned off, the switch is closed simultaneously, so you now have an R-L-C circuit without a battery.

Problem 1

Write the Characteristic equation for the ODE for $t \geq 0$.

$$Q = e^{rt} \implies Lr^2 + Rr + \frac{r}{C} = 0$$

Problem 2

What are the "mass", "damping" and "spring constant" analogues in this system? Does this make intuitive sense?

Judging from the derivative levels of Q, the inductance L is similar to mass, the resistance R is similar to damping and the inverse capacitance $\frac{1}{C}$ is similar to the spring constant.

Problem 3

Determine the relationship between L, R and C when the system is "critically damped".

$$R^2 - 4\frac{L}{C} = 0 \implies R = 2\sqrt{\frac{L}{C}}$$

Problem 4

If $L = 1$, $C = 1$ the equation becomes $Q'' + RQ' + Q = 0$. At $t = 0$, the charge is $Q(0) = 2C$ and the current is zero. We want to explore how fast the amount of charge becomes "negligible" depending on the resistance. More specifically, we want to find the time τ such that $|Q(\tau)| < 0.01$ for all $t > \tau$. In the following you will need to determine the particular solution for different values of R.

$$Q'' + RQ' + Q = 0 \implies Q = e^{rt} \implies r^2 + Rr + 1 = 0 \implies r = \frac{-R \pm \sqrt{R^2 - 4}}{2}$$

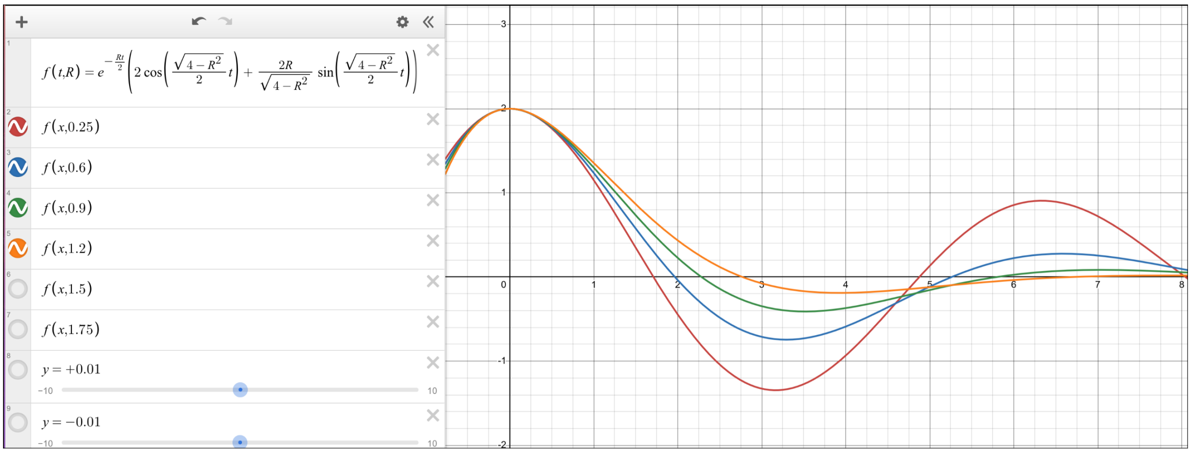
$$\therefore Q = e^{-Rt/2}(c_1 \cos(\omega t) + c_2 \sin(\omega t)),\ \omega = \sqrt{4 - R^2}/2$$

$$\therefore I = Q' = e^{-Rt/2}(\omega c_2 \cos(\omega t) - \omega c_1 \sin(\omega t)) - \frac{R}{2}e^{-Rt/2}(c_1 \cos(\omega t) + c_2 \sin(\omega t))$$

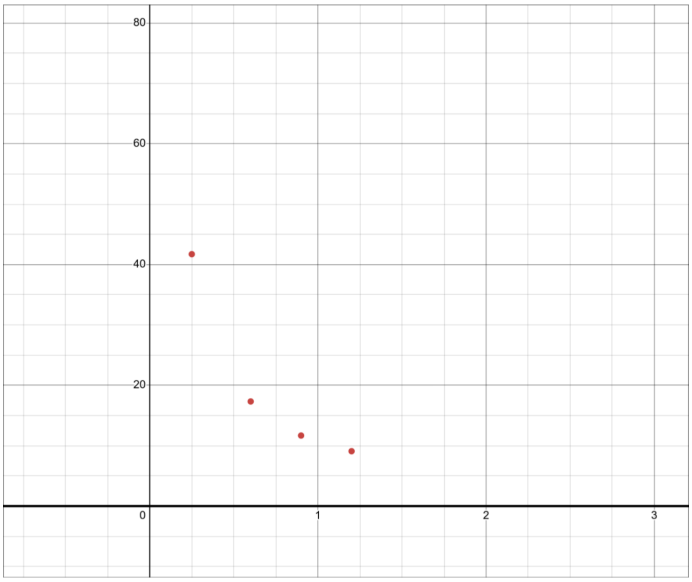
$c_1 = 2, \omega c_2 - \frac{R}{2}c_1 = 0 \implies c_2 = \frac{2R}{\sqrt{R^2 - 4}}$

a) For $R = 0.25\Omega$ estimate τ from the plot of the solution. Use a computer plotter, e.g. Desmos.

b) Do the same for $R = 0.6\Omega$, $R = 0.9\Omega$ and $R = 1.2\Omega$.



c) Plot R vs τ using these 4 points.



d) Now perform part a) for the values of $R = 1.5\Omega$, $R = 1.75\Omega$. $R = 2\Omega$ and add these to the plot.
Solve ODE for $R = 2$. $Q = e^{rt}$, $r = \frac{-R}{2}$, therefore $Q = c_1 e^{-t} + c_2 t e^{-t}$. $c_1 = c_2 = 2$.

R	
0.25	41.71517
0.6	17.33614
0.9	11.68258
1.2	9.09743
1.5	7.18388
1.75	5.0336
2	7.43013

e) Note how τ decreases initially with increasing R and then starts to increase. How would you explain this?
$R = 2\Omega$ is the critically damped situation, where the time it takes to reach a relatively constant value is the fastest, without oscillations. After the critically damped cituation the time to stabilize is longer.