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CS105 Ordinary Differential Equations Section B - Homework 6

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Problem 1: Method of undetermined coefficients

Given an ODE of the form $y'' + Ay' + By = g(t)$ we obtained the particular solution for $g(t) = ke^{at}$, where a is not one of the roots of the characteristic equation, using the method of undetermined coefficients. In this problem, I want you to use the method of undetermined coefficients to obtain the particular solution when $g(t) = kt^2$. Assume an ansatz of the form $y_p(t) = at^2 + bt + c$. To get full points, all possible cases for the constants A, B , including when $B = 0$, must be discussed.

General solution:

$$y'' + Ay' + By = 0 \implies r^2 + Ar + B = 0 \implies r = \frac{-A \pm \sqrt{A^2 - 4B}}{2}$$

Case 1: $A^2 - 4B > 0$.

$$y_c = c_1 e^{\frac{-A+\sqrt{A^2-4B}}{2}} + c_2 e^{\frac{-A-\sqrt{A^2-4B}}{2}}$$

Case 2: $A^2 - 4B = 0$.

$$y_c = c_1 e^{\frac{-A}{2}} + c_2 t e^{\frac{-A}{2}}$$

Case 3: $A^2 - 4B < 0$.

$$y_c = e^{\frac{-A}{2}} \left(c_1 \cos\left(t\sqrt{B - 4A^2}\right) + c_2 \sin\left(t\sqrt{B - 4A^2}\right)\right)$$

Case 4: $B = 0$.

$$y_c = c_1 e^{\frac{-A+\sqrt{A^2-4B}}{2}} + c_2 e^{\frac{-A-\sqrt{A^2-4B}}{2}}$$

Particular solution:
Case 1: $A, B \neq 0$. Take $y_p(t) = at^2 + bt + c$:

$$2a + 2Aat + Ab + Bat^2 + Bbt + Bc = kt^2$$

$$\begin{cases} k = Ba \\ 2Aa + Bb = 0 \\ 2a + Ab + Bc = 0 \end{cases} \overset{A,B\neq 0}{\implies} \begin{cases} a = \frac{k}{B} \\ b = -\frac{B^2}{2Ak} \\ c = \frac{B^2}{2k} - \frac{2k}{B^2} \end{cases} \implies y_p = \frac{k}{B}t^2 - \frac{-B^2}{2Ak}t + \frac{B^2}{2k} - \frac{2k}{B^2}$$

Case 2: $A = 0, B \neq 0$.

$$a = \frac{k}{B}, b = 0, c = -\frac{2k}{B^2} \implies y_p = \frac{k}{B}t^2 - \frac{2k}{B^2}$$

Case 3: $A \neq 0, B = 0$. Take $y_p(t) = at^3 + bt^2 + ct + d$:

$$6at + 2b + 3Aat^2 + 2Abt + Ac = kt^2$$

$$\begin{cases} 3Aa = k \\ 6a + 2Ab = 0 \\ 2b + Ac = 0 \end{cases} \implies \begin{cases} a = \frac{k}{3A} \\ b = -\frac{k}{A^2} \\ c = \frac{2k}{A^3} \end{cases} \implies y_p = \frac{k}{3A}t^3 - \frac{k}{A^2}t^2 + \frac{2k}{A^3}t + d, \text{ d is a constant}$$

Case 4: $A = B = 0$:

$$y'' = kt^2 \implies y = \frac{kt^4}{12} + c_1 t + c_2$$

Problem 2: Method of variation of parameters

Find the particular solution of the general ODE $y'' + Ay' + By = g(t)$, where the characteristic equation has a single root (denoted by r) and $g(t) = ke^{rt}$, using the method variation of parameters. Then, write the general solution.

$$r^2 + Ar + B = 0 \implies r = \frac{-A \pm \sqrt{A^2 - 4B}}{2} = -\frac{A}{2}$$

$$y_c = c_1 e^{-\frac{At}{2}} + c_2 t e^{-\frac{At}{2}}, g = ke^{-\frac{At}{2}}$$

$$W(t) = \begin{vmatrix} e^{-\frac{At}{2}} & te^{-\frac{At}{2}} \\ -\frac{A}{2}e^{-\frac{At}{2}} & (1 - \frac{At}{2})e^{-\frac{At}{2}} \end{vmatrix} = e^{-At}$$

$$W_1(t) = \begin{vmatrix} 0 & te^{-\frac{At}{2}} \\ 1 & (1 - \frac{At}{2})e^{-\frac{At}{2}} \end{vmatrix} = -te^{-\frac{At}{2}} \text{ and } W_2(t) = \begin{vmatrix} e^{-\frac{At}{2}} & 0 \\ -\frac{A}{2}e^{-\frac{At}{2}} & 1 \end{vmatrix} = e^{-\frac{At}{2}}$$

$$y_p(t) = e^{-\frac{At}{2}} \int \frac{ke^{-\frac{At}{2}} \cdot (-te^{-\frac{At}{2}})}{e^{-At}} \, \mathrm{d}t + te^{-\frac{At}{2}} \int \frac{ke^{-\frac{At}{2}} \cdot e^{-\frac{At}{2}}}{e^{-At}} \, \mathrm{d}t$$

The constants from the integration are discarded, as they form part of the solution for the homogeneous equation, which when plugged in will give 0.

$$\therefore y_p(t) = -ke^{-\frac{At}{2}} \int t \, \mathrm{d}t + kte^{-\frac{At}{2}} \int \mathrm{d}t = \frac{t^2 e^{\frac{-At}{2}}}{2}$$

Problem 3: Swinging pendulum

I promised I would discuss the motion of the pendulum during the lectures, but never found time. No worries, introducing new concepts or models during the HW is always a good second option.

We are going to use force balance relations in 2D to derive the equations of motion for a point mass hanging on an infinitely thin completely rigid wire from an unmovable ceiling. Though this might seem very idealistic, this is not a bad approximation for an actual mass hanging from an ordinary wire.

Consider the following picture of the pendulum at an instantaneous position, where it has reached an angle θ with respect to the vertical axis. Denote the mass by m, and the length of the wire by R.

As you can see, there are two forces acting on it at any given instance. The force of gravity, given by $F_g = mg$ that is always pointing down, and the force of tension in the wire, T , that is always pointing towards the hanging point in the ceiling. T changes in magnitude and direction during the motion.

Since the mass on the pendulum is constrained to move along a circular arc, the effective force causing the instantaneous change of the angle can only point along the line that is perpendicular to the wire (the violet colored arrow). There is another effective force, perpendicular to the violet arrow, and pointing up along the wire. That force is responsible for keeping the motion coefficients constrained to the circular arc, but does not contribute to the ODE for the angle θ.

a) Using trigonometry find the magnitude of the violet arrow.

$$F = -mg\sin\theta$$

b) The arc length from the relaxed hanging position, denoted by l, is related to θ by the equation $l = R\theta$. The instantaneous acceleration of the mass along the arc is given by l''. Using the relationship $l = R\theta$ and the effective force you found in part (a) derive the ODE for the angle θ. The quantity $\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}$ is called the angular acceleration.

$$l'' = (R\theta)'' = R\theta'' = \frac{F}{a} = -g\sin\theta \implies \theta'' + \frac{g}{R}\sin\theta = 0$$

c) Notice that the ODE you found is a 2nd order non-linear ODE. While it has exact solutions, they are very complicated. So even the motion of an idealized simple swinging pendulum is given by a very complicated function! If θ is small, it is very common to make the approximation $\sin\theta \approx \theta$, which comes from the Taylor series expansion of the sine. This is called the small angle approximation. Write down the ODE after the approximation.

Using the Fundamental Theorem of Engineering, we get the ODE $\theta'' + \frac{g}{R}\theta = 0$

d) Obtain the general solution for this ODE. Find the solution with the IC $\theta(0) = \frac{\pi}{10}$, $\theta'(0) = 0$.

$$r^2 + \frac{g}{R} = 0 \implies r = \pm\sqrt{\frac{g}{R}}i \implies \theta(t) = a_1 e^{r_1 t} + a_2 e^{r_2 t}$$

$$= a_1\left(\cos\left(\frac{g}{R}t\right) + i\sin\left(\frac{g}{R}t\right)\right) + a_2(\cos\left(\frac{g}{R}t\right) - i\sin\left(\frac{g}{R}t\right)) = c_1\cos\left(\frac{g}{R}t\right) + c_2\sin\left(\frac{g}{R}t\right)$$

$$\theta(0) = \frac{\pi}{10} \implies c_1 = \frac{\pi}{10}, \theta'(0) = 0 \implies c_2 = 0 \implies \theta(t) = \frac{\pi}{10}\cos\left(\frac{g}{R}t\right)$$

e) Now, assume there is air resistance that acts as $-b\theta'$. Add this resistance force to the ODE you obtained in part (c). Find the general solution for this ODE. Assume the resistance is very small, so the discriminant of the characteristic equation is negative.

$$\theta'' + b\theta' + \frac{g}{R}\theta = 0 \implies r^2 + br + \frac{g}{R} = 0 \implies r = \frac{-b \pm \sqrt{b^2 - 4\frac{g}{R}}}{2}$$

Since $b^2 - 4\frac{g}{R} < 0$, we can consider $\left(4\frac{g}{R} - b^2\right)\cdot(-1)$ and take the -1 out of the square root as i.

$$r = \frac{-b}{2} \pm i\sqrt{\frac{g}{R} - \frac{b^2}{4}} \implies \theta(t) = e^{-\frac{b}{2}t}\left(c_1\cos\left(t\sqrt{\frac{g}{R} - b^2 4}\right) + c_2\sin\left(t\sqrt{\frac{g}{R} - \frac{b^2}{4}}\right)\right)$$

Problem 4: Forced Pendulum Motion

In the last part of the last problem you should have obtained the ODE . Notice the similarity with the damped spring-mass system. In the latter, the restorative force by the spring was associated with a constant k, here the analogous constant is given by $\frac{g}{R}$. Thus, the smaller the length of the pendulum, the bigger the restorative force. Of course, this is only true when the angle is small. Does this observation agree with your experience?

Now assume an external force $A\cos\omega t$ is applied to the angle. This can be accomplished by attaching an electromagnetic rotor to the hanging point that applies an oscillating torque to the rigid wire. The resulting ODE is $\theta'' + b\theta' + \frac{g}{R}\theta = A\cos\omega t$.

a) Obtain the particular solution for this ODE. This solution represents the steady-state (long-term) response of the pendulum to the driving force.

Assume $y_p = C_1\cos(\omega t) + C_2\sin(\omega t)$.

$$-\omega^2 C_1\cos(\omega t) - \omega^2 C_2\sin(\omega t) - \omega b C_1\sin(\omega t) + \omega b C_2\cos(\omega t) + \frac{g}{R}C_1\cos(\omega t) + \frac{g}{R}C_2\sin(\omega t) = A\cos\omega t$$

$$\begin{cases} -C_1\omega^2 + bC_2\omega + \frac{g}{R}C_1 = A \\ -C_2\omega^2 - bC_1\omega + \frac{g}{R}C_2 = 0 \end{cases} \implies \begin{cases} C_1(\frac{g}{R} - \omega^2) + C_2(b\omega) = A \\ C_1(-b\omega) + C_2(\frac{g}{R} - \omega^2) = 0 \end{cases}$$

$$\omega_0 := \sqrt{\frac{g}{R}} \begin{cases} C_1(-b\omega) = -C_2(\omega_0^2 - \omega^2) \\ C_2\frac{(\omega_0^2-\omega^2)^2}{b\omega} + C_2(b\omega) = A \end{cases} \implies \begin{cases} C_2 = \frac{Ab\omega}{b^2\omega^2+(\omega_0^2-\omega^2)^2} \\ C_1 = \frac{A(\omega_0^2-\omega^2)}{b^2\omega^2+(\omega_0^2-\omega^2)^2} \end{cases}$$

$$\therefore y_p = \frac{A(\omega_0^2 - \omega^2)}{b^2\omega^2 + (\omega_0^2 - \omega^2)^2}\cos(\omega t) + \frac{Ab\omega}{b^2\omega^2 + (\omega_0^2 - \omega^2)^2}\sin(\omega t)$$

b) Express the steady-state solution in the form $C\cos(\omega t - \varphi)$. Simplify the expressions for C and φ.

$$y_p = \frac{A}{\sqrt{b^2\omega^2 + (\omega_0^2 - \omega^2)^2}}\cos\left(\omega t - \arctan\left(\frac{Ab\omega}{\omega_0^2 - \omega^2}\right)\right)$$

c) What is the value of ω for which the amplitude of the forced swing is maximized?

$$\omega = \omega_0 = \sqrt{\frac{g}{R}} \implies y = \frac{A}{b\omega}\cos\left(\omega t - \frac{\pi}{2}\right)$$