Digital version

CS105 Ordinary Differential Equations - Homework 10

Mher Saribekyan A09210183

April 28, 2025

Page 344: Problems 2,4,19

In each of Problems 1 through 4 transform the given equation into a system of first order equations.

2) $u'' + 0.5u' + 2u = 3\sin t$

$$\begin{cases} x_1' = x_2 \\ x_2' + 0.5x_2 + 2x_1 = 3\sin t \end{cases}$$

4) $u'''' - u = 0$

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_4 \\ x_4' = x_1 \end{cases}$$

19) Consider the circuit shown in Figure 7.1.4. Use the method outlined in Problem 18 to show that the current I through the inductor and the voltage V across the capacitor satisfy the system of differential equations

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -I - V, \ \frac{\mathrm{d}V}{\mathrm{d}t} = 2I - V$$

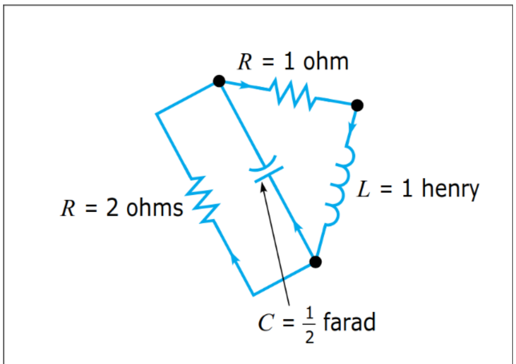


FIGURE 7.1.4 The circuit in Problem 19.

$$\begin{cases} V_{R2} + V_C = 0 \\ V_{R1} + V_L + V_C = 0 \\ I_1 + I_2 + I_3 = 0 \end{cases} , \begin{cases} R_2 I_1 = V_{R2} \\ CV_C' = -I_2 \\ R_1 I_3 + LI_3' = -V_C \end{cases} \implies \begin{cases} V = -2(0.5V' - I) \\ I + I' = -V \end{cases} \implies \begin{cases} V' = 2I - V \\ I' = -I - V \end{cases}$$

Page 356: Problem 21 c,d

In this problem first write each of the matrices as a "matrix with constant coefficients" times "vector-valued function". Then perform the operations.

If $A(t) = \begin{bmatrix} e^t & 2e^{-t} & e^{2t} \\ 2e^t & e^{-t} & -e^{2t} \\ -e^t & 3e^{-t} & 2e^{2t} \end{bmatrix}$ and $B(t) = \begin{bmatrix} 2e^t & e^{-t} & 3e^{2t} \\ -e^t & 2e^{-t} & e^{2t} \\ 3e^t & -e^{-t} & -e^{2t} \end{bmatrix}$, find

c) $\dfrac{\mathrm{d}A}{\mathrm{d}t}$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \begin{bmatrix} e^t & -2e^{-t} & 2e^{2t} \\ 2e^t & -e^{-t} & -2e^{2t} \\ -e^t & -3e^{-t} & 4e^{2t} \end{bmatrix}$$

d) $\int_0^1 A(t)\,\mathrm{d}t$

$$\int_0^1 A(t)\,\mathrm{d}t = \begin{bmatrix} (e-1) & -2(e^{-1}-1) & \frac{e^2-1}{2} \\ 2(e-1) & -(e^{-1}-1) & -\frac{e^2-1}{2} \\ -(e-1) & -3(e^{-1}-1) & e^2-1 \end{bmatrix} = (e-1)\begin{bmatrix} 1 & 2e^{-1} & 0.5(e+1) \\ 2 & e^{-1} & -0.5(e+1) \\ -1 & 3e^{-1} & (e+1) \end{bmatrix}$$

Page 365: Problems 14, 30

14) Let $\mathbf{x}^{(1)}(t) = \begin{bmatrix} e^t \\ te^t \end{bmatrix}$, $\mathbf{x}^{(2)}(t) = \begin{bmatrix} 1 \\ t \end{bmatrix}$.
Show that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly dependent at each point in the interval $0 \le t \le 1$.

$$\forall a \in [0,1],\ x^{(1)}(a) = e^a \begin{bmatrix} 1 \\ a \end{bmatrix}, x^{(2)} = \begin{bmatrix} 1 \\ a \end{bmatrix} \implies x^{(1)} = e^a x^{(2)} \implies \text{linearly dependent}$$

Nevertheless, show that $\mathbf{x}^{(1)}(t)$ and $\mathbf{x}^{(2)}(t)$ are linearly independent on $0 \le t \le 1$.

$$\text{Assume dependent} \implies \forall t \in [0,1],\ x^{(1)}(t) = cx^{(2)}(t),\ c \in \mathbb{C} \implies \begin{cases} e^t = c \\ te^t = tc \end{cases} \implies \text{impossible}$$

$$\therefore x^{(1)}(t) \text{ and } x^{(2)}(t) \text{ are linearly independent}$$

30) Prove that $\lambda = 0$ is an eigenvalue of A if and only if A is singular.

$$A := \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{21} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

If λ is an eigenvalue of A, then:

$$\begin{vmatrix} (a_{11}-\lambda) & a_{12} & \dots & a_{1n} \\ a_{21} & (a_{21}-\lambda) & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & (a_{nn}-\lambda) \end{vmatrix} = 0$$

If $\lambda = 0$, the above matrix is the same as the matrix A, therefore $\det(A)$ has to be zero.

Find eigenvalues of the matrix A:

$$\begin{vmatrix} (a_{11}-\lambda) & a_{12} & \dots & a_{1n} \\ a_{21} & (a_{21}-\lambda) & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & (a_{nn}-\lambda) \end{vmatrix} = 0$$

If the determinant of matrix A is 0, then:

$$\begin{vmatrix} (a_{11}-0) & a_{12} & \dots & a_{1n} \\ a_{21} & (a_{21}-0) & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & (a_{nn}-0) \end{vmatrix} = \det(A) = 0 \implies \lambda_1 = 0$$

Therefore $\lambda = 0$ is an eigenvalue of a singularm matrix.

Page 381: Problems 7,17

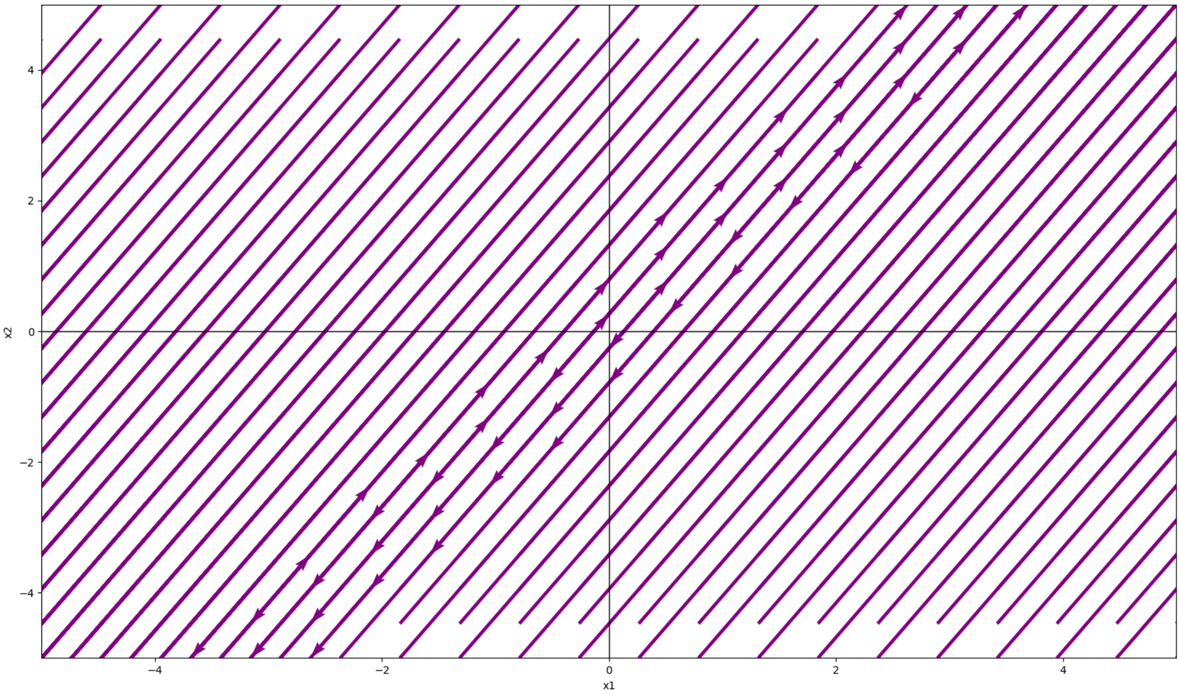
In each of Problems 7 and 8 find the general solution of the given system of equations. Also draw a direction field and a few of the trajectories. In each of these problems the coefficient matrix has a zero eigenvalue. As a result, the pattern of trajectories is different from those in the examples in the text.
$\mathbf{x}' = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \mathbf{x}$

$$\begin{vmatrix} (4-\lambda) & -3 \\ 8 & (-6-\lambda) \end{vmatrix} = 0 \implies (4-\lambda)(-6-\lambda) + 24 = \lambda^2 + 2\lambda = 0 \implies \lambda_1 = 0, \lambda_2 = -2$$

$$\lambda_1 = 0 : \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \mathbf{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\lambda_2 = 2 : \begin{bmatrix} 6 & -3 \\ 8 & -4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\therefore \mathbf{x} = c_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-2t}$$



In each of Problems 15 through 18 solve the given initial value problem. Describe the behavior of the solution as $t \to \infty$.

$\mathbf{x}' = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{bmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

$$\lambda_1 = 1, \mathbf{v}_1 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}, \ \lambda_2 = 2, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \lambda_3 = 3, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{3t}$$

$$\begin{cases} c_2 + c_3 = 2 \\ -2c_1 + c_2 + 3c_3 = 0 \\ c_1 + c_3 = 1 \end{cases} \implies \begin{cases} c_2 = 2 - c_3 \\ -2c_1 + 2 - c_3 + 3c_3 = 0 \\ c_1 + c_3 = 1 \end{cases} \implies \begin{cases} c_1 = 1 \\ c_2 = 2 \\ c_3 = 0 \end{cases}$$

$$\therefore \mathbf{x}(t) = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} e^{2t}$$

As $t \to \infty$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, x_1, x_2, x_3 \to \infty$.

Page 391: Problem 25

25. Consider the electric circuit shown in Figure 7.6.5. Suppose that R1 = R2 = 4 ohms, C = 1/2 farad, and L = 8 henrys.

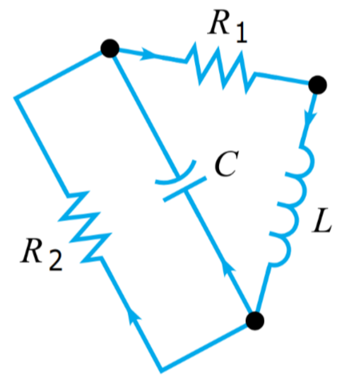


FIGURE 7.6.5 The circuit in Problem 25.

a) Show that this circuit is described by the system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} I \\ V \end{bmatrix}$$

where I is the current through the inductor and V is the voltage drop across the capacitor. Hint: See Problem 18 of Section 7.1.

$$\begin{cases} V_{R2} + V_C = 0 \\ V_{R1} + V_L + V_C = 0 \\ I_1 + I_2 + I_3 = 0 \end{cases}, \begin{cases} R_2 I_1 = V_{R2} \\ C V_C' = -I_2 \\ R_1 I_3 + L I_3' = -V_C \end{cases} \implies \begin{cases} V = -4(\frac{1}{2}V' - I) \\ 4I + 8I' = -V \end{cases} \implies \begin{cases} V' = 2I - \frac{1}{2}V \\ I' = -\frac{1}{2}I - \frac{1}{8}V \end{cases}$$

$$\therefore \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{8} \\ 2 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} I \\ V \end{bmatrix}$$

(b) Find the general solution of Eqs. (i) in terms of real-valued functions.

$$\lambda_1 = -\frac{1}{2} - \frac{i}{2}, \mathbf{v}_1 = \begin{bmatrix} i \\ 4 \end{bmatrix}, \ \lambda_2 = -\frac{1}{2} + \frac{i}{2}, \mathbf{v}_2 = \begin{bmatrix} i \\ -4 \end{bmatrix}$$

$$e^{\lambda_1} = e^{-\frac{1}{2}t} + \cos(t/2) - i\sin(t/2)$$

$$e^{\lambda_2} = e^{-\frac{1}{2}t} + \cos(t/2) + i\sin(t/2)$$

$$\begin{bmatrix} I \\ V \end{bmatrix} = a_1 e^{-\frac{1}{2}t}\begin{bmatrix} i\cos(t/2) + \sin(t/2) \\ 4\cos(t/2) - 4i\sin(t/2) \end{bmatrix} + a_2 e^{-\frac{1}{2}t}\begin{bmatrix} i\cos(t/2) - \sin(t/2) \\ -4\cos(t/2) - 4i\sin(t/2) \end{bmatrix}$$

$$\begin{cases} c_1 := a_1 i + a_2 i \\ c_2 := a_1 - a_2 \end{cases} \implies \begin{bmatrix} I \\ V \end{bmatrix} = c_1 e^{-\frac{1}{2}t}\begin{bmatrix} \cos(t/2) \\ 4\sin(t/2) \end{bmatrix} + c_2 e^{-\frac{1}{2}t}\begin{bmatrix} \sin(t/2) \\ -4\cos(t/2) \end{bmatrix}$$

(c) Find I (t) and V (t) if I (0) = 2 amperes and V (0) = 3 volts.

$$2 = c_1,\ 3 = -4c_2 \implies \begin{bmatrix} I \\ V \end{bmatrix} = 2e^{-\frac{1}{2}t} \begin{bmatrix} \cos(t/2) \\ 4\sin(t/2) \end{bmatrix} - \frac{3}{4}e^{-\frac{1}{2}t} \begin{bmatrix} \sin(t/2) \\ -4\cos(t/2) \end{bmatrix}$$

(d) Determine the limiting values of I (t) and V (t) as $t \to \infty$. Do these limiting values depend on the initial conditions?

For both I and V, as $t \to \infty$, both values approach 0, as the term $e^{-\frac{1}{2}t}$ approaches 0, no matter the initial conditions.