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CS105 Ordinary Differential Equations - Homework 11

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Page 391 - Problems 16,17,25,26

16. Figure 5.5.13 shows two railway cars with a buffer spring. We want to investigate the transfer of momentum that occurs after car 1 with initial velocity v_0 impacts car 2 at rest. The analog of Eq. (18) in the text is

$$\mathbf{x}'' = \begin{bmatrix} -c_1 & c_1 \\ c_2 & -c_2 \end{bmatrix} \mathbf{x}$$

with $c_i = k/m_i$ for $i = 1, 2$. Show that the eigenvalues of the coefficient matrix $\mathbf{A}$ are $\lambda_1 = 0$ and $\lambda_2 = -c_1 - c_2$, with associated eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $\mathbf{v}_2 = \begin{bmatrix} c_1 & -c_2 \end{bmatrix}^T$.

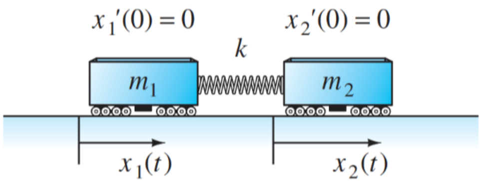

FIGURE 5.5.13. The two railway cars of Problems 16 through 19.

$$\begin{vmatrix} -c_1 - \lambda & c_1 \\ c_2 & -c_2 - \lambda \end{vmatrix} = (-c_1 - \lambda)(-c_2 - \lambda) - c_1 c_2 = \lambda^2 + \lambda(c_1 + c_2) = \lambda(\lambda + c_1 + c_2) = 0$$

$$\begin{cases} \lambda_1 = 0 \\ \lambda_2 = -c_1 - c_2 \end{cases} \implies \begin{cases} A - \lambda_1 I = \begin{bmatrix} -c_1 & c_1 \\ c_2 & -c_2 \end{bmatrix} \\ A - \lambda_2 I = \begin{bmatrix} c_2 & c_1 \\ c_2 & c_1 \end{bmatrix} \end{cases} \implies \begin{cases} \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \mathbf{v}_2 = \begin{bmatrix} c_1 \\ -c_2 \end{bmatrix} \end{cases}$$

17. If the two cars of Problem 16 both weigh 16 tons (so that $m_1 = m_2 = 1000$ (slugs)) and $k = 1$ ton/ft (that is, 2000 lb/ft), show that the cars separate after $\pi/2$ seconds, and that $x_1'(t) = 0$ and $x_2'(t) = v_0$ thereafter. Thus the original momentum of car 1 is completely transferred to car 2.

$$c_1 = c_2 = \frac{1}{16} \implies \begin{cases} \lambda_0 = 0 \\ \lambda_1 = -\frac{1}{8} \end{cases} \implies \begin{cases} \mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{cases}$$

$$\begin{cases} \mathbf{x}_0(t) = (a_0 + b_0 t)\mathbf{v}_0 \\ \mathbf{x}_1(t) = (a_1 \cos\left(\frac{1}{2\sqrt{2}}t\right) + b_1 \sin\left(\frac{1}{2\sqrt{2}}t\right))\mathbf{v}_1 \end{cases}$$

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mathbf{x}'(0) = \begin{bmatrix} v_0 \\ 0 \end{bmatrix} \implies \begin{cases} a_0 + a_1 = 0 \\ a_0 - a_1 = 0 \\ b_0 + \frac{1}{2\sqrt{2}}b_1 = v_0 \\ b_0 - \frac{1}{2\sqrt{2}}b_1 = 0 \end{cases} \implies \begin{cases} a_0 = a_1 = 0 \\ b_0 = \frac{v_0}{2} \\ b_1 = v_0\sqrt{2} \end{cases}$$

$$\mathbf{x}(t) = \frac{v_0}{2}t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + v_0\sqrt{2}\sin\left(\frac{1}{2\sqrt{2}}t\right) \begin{bmatrix} 1 \\ -1 \end{bmatrix} = v_0 \begin{bmatrix} t/2 + \sqrt{2}\sin\left(\frac{1}{2\sqrt{2}}t\right) \\ t/2 - \sqrt{2}\sin\left(\frac{1}{2\sqrt{2}}t\right) \end{bmatrix}$$

$$\mathbf{x}'(t) = \frac{v_0}{2} \begin{bmatrix} 1 + \cos\left(\frac{1}{2\sqrt{2}}t\right) \\ 1 - \cos\left(\frac{1}{2\sqrt{2}}t\right) \end{bmatrix}$$

$$\cos\left(\frac{1}{2\sqrt{2}}t\right) = 1 \implies t_1 = 2\pi\sqrt{2} \implies \mathbf{x}'(t_1) = \begin{bmatrix} 0 \\ v_0 \end{bmatrix}$$

25. Suppose that $m = 75$ slugs (the car weighs 2400 lb), $L_1 = 7$ ft, $L_2 = 3$ ft (it's a rear-engine car), $k_1 = k_2 = 2000$ lb/ft, and $I = 1000$ ft·lb·s^2. Then the equations in (40) take the form

$$75x'' + 4000x - 8000\theta = 0,$$

$$1000\theta'' - 8000x + 116{,}000\theta = 0.$$

(a) Find the two natural frequencies ω_1 and ω_2 of the car.
(b) Now suppose that the car is driven at a speed of v feet per second along a washboard surface shaped like a sine curve with a wavelength of 40 ft. The result is a periodic force on the car with frequency $\omega = 2\pi v/40 = \pi v/20$. Resonance occurs when with $\omega = \omega_1$ or $\omega = \omega_2$. Find the corresponding two critical speeds of the car (in feet per second and in miles per hour).

$$\begin{bmatrix} 75 & 0 \\ 0 & 1000 \end{bmatrix} \mathbf{x}'' = \begin{bmatrix} -4000 & 8000 \\ 8000 & -116000 \end{bmatrix} \mathbf{x}$$

26. Suppose that $k_1 = k_2 = k$ and $L_1 = L_2 = \frac{1}{2}L$ in Fig. 5.5.14 (the symmetric situation). Then show that every free oscillation is a combination of a vertical oscillation with frequency

$$\omega_1 = \sqrt{2k/m}$$

and an angular oscillation with frequency

$$\omega_2 = \sqrt{kL^2/(2I)}.$$

$$\mathbf{x}'' = \begin{bmatrix} -\frac{2k}{m} & 0 \\ 0 & -\frac{kL^2}{2l} \end{bmatrix} \mathbf{x}$$

$$\lambda_1 = -\frac{2k}{m} \text{ and } \lambda_2 = -\frac{kL^2}{2l} \implies \omega_1 = \sqrt{\frac{2k}{m}} \text{ and } \omega_2 = \sqrt{\frac{kL^2}{2l}}$$

Page 419 - Problems 1, 9, 35

Find a fundamental matrix of each of the systems in Problems 1 through 8, then apply Eq. (8) to find a solution satisfying the given initial conditions.

1. $\mathbf{x}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = \lambda^2 - 4\lambda + 3 = 0 \implies \lambda = 2 \pm 1$$

$$\lambda_1 = 1,\ \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ \lambda_2 = 3,\ \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \mathbf{\Phi}(t) = \begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix}$$

$$\mathbf{\Phi}^{-1}(t) = \frac{1}{2e^{4t}} \begin{bmatrix} e^{3t} & -e^{3t} \\ e^t & e^t \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^{-t} & -e^{-t} \\ e^{-3t} & e^{-3t} \end{bmatrix}$$

$$\mathbf{x}(t) = \frac{1}{2} \begin{bmatrix} e^t & e^{3t} \\ -e^t & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^t + e^{3t} & -e^t + e^{3t} \\ -e^t + e^{3t} & e^t + e^{3t} \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 5e^t + e^{3t} \\ -5e^t + e^{3t} \end{bmatrix}$$

Compute the matrix exponential $e^{\mathbf{A}t}$ for each system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ given in Problems 9 through 20.

9. $x_1' = 5x_1 - 4x_2,\ x_2' = 2x_1 - x_2$

$$\mathbf{x}'(t) = \begin{bmatrix} 5 & -4 \\ 2 & -1 \end{bmatrix} \mathbf{x}(t)$$

$$(5-\lambda)(-1\lambda) + 8 = 0 \implies \begin{cases} \lambda_1 = 3, \mathbf{v}_1 = \begin{bmatrix} 2 & 1 \end{bmatrix}^T \\ \lambda_2 = 1, \mathbf{v}_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T \end{cases}$$

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{t} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2e^{3t} - e^t & -2e^{3t} + 2e^t \\ e^{3t} - e^t & -e^{3t} + 2e^t \end{bmatrix}$$

Apply Theorem 3 to calculate the matrix exponential $e^{\mathbf{A}t}$ for each of the matrices in Problems 35 through 40.

35. $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 0 & 3 \end{bmatrix}$ **36.** $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

$$(3-\lambda)^2 = 0 \implies \lambda = 3, \mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(A - 3I)^2 \mathbf{u} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{x}_1(t) = \begin{bmatrix} e^{3t} \\ 0 \end{bmatrix}, \ \mathbf{x}_2(t) = \begin{bmatrix} 4e^{3t}t \\ e^{3t} \end{bmatrix} \implies \mathbf{\Phi}(t) = \begin{bmatrix} e^{3t} & 4te^{3t} \\ 0 & e^{3t} \end{bmatrix}$$

$$\mathbf{\Phi}(0)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \implies e^{\mathbf{A}t} = \begin{bmatrix} e^{3t} & 4te^{3t} \\ 0 & e^{3t} \end{bmatrix}$$

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Apply the method of undetermined coefficients to find a particular solution of each of the systems in Problems 1 through 14. If initial conditions are given, find the particular solution that satisfies these conditions. Primes denote derivatives with
2 *respect to t.*

8. $x' = x - 5y + 2\sin t$, $y' = x - y - 3\cos t$

$$\mathbf{x}'(t) = \begin{bmatrix} 1 & -5 \\ 1 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2\sin t \\ -3\cos t \end{bmatrix}$$

$$\mathbf{x}_p(t) := \mathbf{a}\sin(t) + \mathbf{b}\cos(t) = \begin{bmatrix} a_1 \sin(t) + b_1 \cos(t) \\ a_2 \sin(t) + b_2 \cos(t) \end{bmatrix}$$

$$\begin{bmatrix} a_1 \cos(t) - b_1 \sin(t) \\ a_2 \cos(t) - b_2 \sin(t) \end{bmatrix} = \begin{bmatrix} a_1 \sin(t) + b_1 \cos(t) - 5\left(a_2 \sin(t) + b_2 \cos(t)\right) \\ a_1 \sin(t) + b_1 \cos(t) - a_2 \sin(t) - b_2 \cos(t) \end{bmatrix} + \begin{bmatrix} 2\sin t \\ -3\cos t \end{bmatrix}$$

$$\begin{cases} a_1 = b_1 - 5b_2 + 2 \\ -b_1 = a_1 - 5a_2 + 2 \\ a_2 = b_1 - b_2 - 3 \\ -b_2 = a_1 - a_2 \end{cases} \implies \begin{cases} x(t) = \frac{1}{3}(19\cos t) \\ y(t) = \frac{1}{3}(5\cos t + 5\sin t) \end{cases}$$

In Problems 17 through 34, use the method of variation of parameters (and perhaps a computer algebra system) to solve the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t), \quad \mathbf{x}(a) = \mathbf{x}_a.$$

In each problem we provide the matrix exponential $e^{\mathbf{A}t}$ as provided by a computer algebra system.

17. $\mathbf{A} = \begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 60 \\ 90 \end{bmatrix}$, $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$$e^{\mathbf{A}t} = \frac{1}{6}\begin{bmatrix} -e^{-t} + 7e^{5t} & 7e^{-t} - 7e^{5t} \\ -e^{-t} + e^{5t} & 7e^{-t} - e^{5t} \end{bmatrix}$$

$$e^{\mathbf{A}t} = \begin{bmatrix} -e^{-t} + 7e^{5t} & 7e^{-t} - 7e^{5t} \\ -e^{-t} + e^{5t} & 7e^{-t} - e^{5t} \end{bmatrix} \implies e^{-\mathbf{A}t} = \frac{1}{36}\begin{bmatrix} 7e^{-5t} - e^{t} & -7e^{-5t} + e^{t} \\ -e^{-5t} + e^{t} & -e^{-5t} + 7e^{t} \end{bmatrix}$$

$$e^{-\mathbf{A}t}\mathbf{x}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \int_0^t \frac{1}{36}\begin{bmatrix} 7e^{-5s} - e^{s} & -7e^{-5s} + e^{s} \\ -e^{-5s} + e^{s} & -e^{-5s} + 7e^{s} \end{bmatrix}\begin{bmatrix} 60 \\ 90 \end{bmatrix} \mathrm{d}s = \frac{1}{36}\int_0^t \begin{bmatrix} 7e^{-5s} - e^{s} & -7e^{-5s} + e^{s} \\ -e^{-5s} + e^{s} & -e^{-5s} + 7e^{s} \end{bmatrix}\begin{bmatrix} 60 \\ 90 \end{bmatrix} \mathrm{d}s$$

$$= \frac{1}{36}\begin{bmatrix} 30e^{t} + 42e^{-5t} - 72 \\ 690e^{t} + 30e^{-5t} - 720 \end{bmatrix}$$

$$\mathbf{x}(t) = \frac{1}{36}\begin{bmatrix} -e^{-t} + 7e^{5t} & 7e^{-t} - 7e^{5t} \\ -e^{-t} + e^{5t} & 7e^{-t} - e^{5t} \end{bmatrix}\begin{bmatrix} 30e^{t} + 42e^{-5t} - 72 \\ 690e^{t} + 30e^{-5t} - 720 \end{bmatrix} = \begin{bmatrix} \frac{14e^{-6t} + 378e^{5t} - 414e^{-t} - 385e^{6t} + 407}{3} \\ \frac{14e^{-6t} + 54e^{5t} - 414e^{-t} - 55e^{6t} + 401}{3} \end{bmatrix}$$

$$\therefore x_1(t) = \frac{1}{3}\left(14e^{-6t} + 378e^{5t} - 414e^{-t} - 385e^{6t} + 407\right) x_2(t) = \frac{1}{3}\left(14e^{-6t} + 54e^{5t} - 414e^{-t} - 55e^{6t} + 401\right)$$