Digital version

CS105 Ordinary Differential Equations - Homework 7

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Part 1: Finding the Fourier Series

Consider watching the 3blue1brown video on Fourier series:
`https://www.youtube.com/watch?v=r6sGWTCMz2k`

Problem 1: Find the Fourier series of the following periodic functions:

a) $f(t) = 2\sin\left(t - \frac{\pi}{3}\right)$ (hint: easier than it seems)

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} 2\sin\left(t - \frac{\pi}{3}\right)\mathrm{d}t = 0$$

$$a_m = \frac{1}{\pi}\int_0^{2\pi} 2\sin\left(t - \frac{\pi}{3}\right)\cdot\cos(mt)\,\mathrm{d}t = \frac{1}{\pi}\int_0^{2\pi} \sin\left((m+1)t - \frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3} + (m-1)t\right)\mathrm{d}t$$

$$a_1 = \frac{1}{\pi}\left[-\frac{1}{2}\cos\left(2t - \frac{\pi}{3}\right) - t\sin\left(\frac{\pi}{3}\right)\right]_0^{2\pi} = -\sqrt{3}, a_m = 0, m \neq 1$$

$$b_m = \frac{1}{\pi}\int_0^{2\pi} 2\sin\left(t - \frac{\pi}{3}\right)\cdot\sin(mt)\,\mathrm{d}t = \frac{1}{\pi}\int_0^{2\pi} \cos\left((m+1)t - \frac{\pi}{3}\right) + \cos\left((m-1)t - \frac{\pi}{3}\right)\mathrm{d}t$$

$$b_1 = \frac{1}{\pi}\left[\frac{1}{2}\sin\left(2t - \frac{\pi}{3}\right) + t\cos\left(\frac{\pi}{3}\right)\right]_0^{2\pi} = 1, b_m = 0, m \neq 1$$

$$\therefore f(t) = \sin(t) - \sqrt{3}\cos(t)$$

b) $f(t) = |t|$ in the interval $t \in [-\pi, \pi]$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |t|\,\mathrm{d}t = \pi$$

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} |t|\cos(mt)\,\mathrm{d}t \overset{\text{even}}{=} \frac{2}{\pi}\int_0^{\pi} t\cos(mt)\,\mathrm{d}t = \left[\frac{2}{\pi}t\sin(mt)\right]_0^{\pi} - \frac{2}{m\pi}\int_0^{\pi}\sin(mt)\,\mathrm{d}t$$

$$= \frac{2}{m^2\pi}\left[\cos(mt)\right]_0^{\pi}$$

If m is even, $a_m = 0$. If m is odd, $a_m = -\frac{4}{m^2\pi}$.

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} |t| \sin(mt)\,\mathrm{d}t \overset{\text{odd}}{=} 0$$

$$\therefore f(t) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}\cos((2n+1)t)$$

Recall the square wave function we analyzed during the lecture: $SQ(t) = -1$, for $t \in \left[-\frac{T}{2}, 0\right)$ and $SQ(t) = 1$ for $t \in \left[0, \frac{T}{2}\right)$. Find the Fourier series of

c) $SQ(t-1)$ when $T = 4$

$$a_0 = \frac{1}{2}\int_{-2}^{2} SQ(t-1)\,\mathrm{d}t = -1$$

$$a_m = \frac{1}{2}\int_{-2}^{2} SQ(t-1)\cos(m\pi t)\,\mathrm{d}t = \frac{1}{2}\int_{1}^{2}\cos(m\pi t)\,\mathrm{d}t - \frac{1}{2}\int_{-2}^{1}\cos(m\pi t)\,\mathrm{d}t$$

$$= \frac{1}{2m\pi}\Big(\sin(m\pi) - \sin\Big(\frac{m\pi}{2}\Big) - \sin\Big(\frac{m\pi}{2}\Big) + \sin(-m\pi)\Big) = -\frac{2}{m\pi}\sin\Big(\frac{m\pi}{2}\Big)$$

$$b_m = \frac{1}{2}\int_{-2}^{2} SQ(t-1)\sin(m\pi t)\,\mathrm{d}t = \frac{1}{2}\int_{1}^{2}\sin(m\pi t)\,\mathrm{d}t - \frac{1}{2}\int_{-2}^{1}\sin(m\pi t)\,\mathrm{d}t$$

$$= \frac{1}{2m\pi}\Big(-\cos(m\pi) + \cos\Big(\frac{m\pi}{2}\Big) + \cos\Big(\frac{m\pi}{2}\Big) - \cos(-m\pi)\Big) = \frac{2}{m\pi}\Big(\cos\Big(\frac{m\pi}{2}\Big) - \cos(m\pi)\Big)$$

$$\therefore SQ(t-1) = -\frac{1}{2} + \sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}\sin\left(\frac{\pi n}{2}\right)\cos\left(\frac{\pi n}{2}t\right)\right) + \sum_{n=1}^{\infty}\left(\frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - \cos\left(n\pi\right)\right)\sin\left(\frac{\pi n}{2}t\right)\right)$$

d) $1 + SQ(2\pi t)$ when $T = 2$

$$a_0 = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} 1 + SQ(t)\,\mathrm{d}t = 2$$

$$a_m = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} (1 + SQ(t))\cos\Big(\frac{mt}{2}\Big)\,\mathrm{d}t = 0$$

$$b_m = \frac{1}{2\pi}\int_{-2\pi}^{2\pi} (1 + SQ(t))\sin\Big(\frac{mt}{2}\Big)\,\mathrm{d}t = \frac{1}{\pi}\int_{0}^{2\pi}\sin\Big(\frac{mt}{2}\Big)\,\mathrm{d}t = -\frac{2}{m\pi}\left(\cos(\pi m) - 1\right)$$

$$\therefore 1 + SQ(2\pi t) = 1 - \sum_{n=0}^{\infty}\frac{2}{(2n+1)\pi}\left(\cos\left(\pi\left(2n+1\right)\right) - 1\right)\left(\sin\left(\frac{(2n+1)\,t}{2}\right)\right)$$

Part 2: Solving ODEs

If you have an engineering focus, consider watching the special MIT lecture:
`https://www.youtube.com/watch?v=pRIEYR5JHQA&list=PLEC88901EBADDD980&index=18`

Another physical system that is well described by a 2nd order linear ODE with constant coefficients is the vibration of the trampoline - a beam of solid material (like wood) that has much more width than thickness (imagine the trampoline the swimmers use for diving). The tip of the trampoline satisfies the ODE:

$$y'' + by' + \omega_0^2 y = F(t)$$

where $F(t)$ is the external force, ω_0 is the natural frequency, and b represents the damping. Since the trampoline is made of real material, as it bends there is a build up of internal strains. The strains have a critical threshold after which the trampoline breaks (if it is wood, for example) or bends in a non-reversible way (if it is aluminum, for example). Such non-reversible bending is called plastic deformation, whereas the reversible bending that happens while it obeys the ODE is called elastic deformation.

Assume $F(t) = SQ(t)$, where $SQ(t)$ is the square wave function with period $T = 2s$.

Take $b = 1s^{-1}$, $\omega_0 = 5rad/s$ and assume the maximum amplitude the tip of the trampoline can reach before breaking is $|y_c| = 0.15m$.

a) Use the Fourier series of $F(t)$ to find the Fourier series of the particular solution $y_p(t)$.

   It is trivial that: $F(t) = \sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}\left((-1)^n - 1\right)\sin(n\pi\ t)\right)$. Using the assumption and calculations in HW6 Problem 4, it is obvious that:

$$y_p = \frac{A(\omega_0^2 - \omega^2)}{b^2\omega^2 + (\omega_0^2 - \omega^2)^2}\sin(\omega t) + \frac{Ab\omega}{b^2\omega^2 + (\omega_0^2 - \omega^2)^2}\cos(\omega t)$$

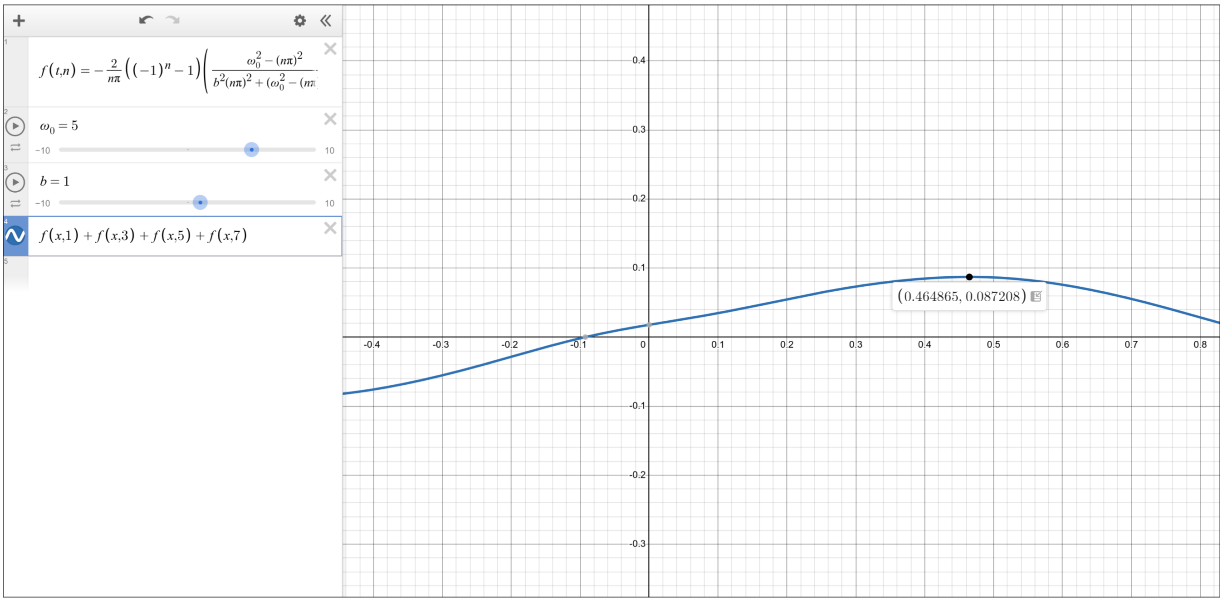
   Therefore, the solution to our equation is:

$$y_p = -\frac{2}{n\pi}\sum_{n=1}^{\infty}\left((-1)^n - 1\right)\left(\frac{\omega_0^2 - (n\pi)^2}{b^2(n\pi)^2 + (\omega_0^2 - (n\pi)^2)^2}\sin(n\pi\ t) + \frac{b(n\pi)}{b^2(n\pi)^2 + (\omega_0^2 - (n\pi)^2)^2}\cos(n\pi\ t)\right)$$

b) Determine the 4 terms in the infinite sum of $y_p(t)$ that contribute the most to the motion (have the highest amplitudes).

   The terms $n = 1, 3, 5, 7$ have the highest contribution to the motion.

c) Use a plotting tool (such as Desmos) to plot the sum of these 4 terms, and determine the maximum value the resulting function reaches.



The maximum value is $y_{\text{max}} = 0.087$.

d) Does the trampoline break from the analysis of just these 4 terms?

Nope, not even close.

Hint: You can use the results from HW 6 problem 4 to shorten your time spent on this problem.