Digital version

ENGS261 Control Systems - Homework 5

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Problem 1: From a transfer function to state space

Consider

$$G(s) = \frac{Y(s)}{U(s)} = \frac{2s^2 + 7s + 15}{s^3 + 6s^2 + 11s + 6}.$$

a) Derive a phase-variable realization

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu, \qquad y = C\mathbf{x} + Du,$$

and identify the phase variables you used.

$$\frac{Y(s)}{U(s)} = \frac{Y(s)}{V(s)} \cdot \frac{V(s)}{U(s)} = (2s^2 + 7s + 15)\frac{1}{s^3 + 6s^2 + 11s + 6}.$$

$$\frac{V(s)}{U(s)} = \frac{1}{s^3 + 6s^2 + 11s + 6} \implies \dddot{v} + 6\ddot{v} + 11\dot{v} + 6v = u$$

Define: $x_1 = v$, $x_2 = \dot{v}$, $x_3 = \ddot{v}$ and $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$.

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -6x_3 - 11x_2 - 6x_1 + u \end{cases} \implies \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$\frac{Y(s)}{V(s)} = 2s^2 + 7s + 15 \implies y = 2\ddot{v} + 7\dot{v} + 15v = 15x_1 + 7x_2 + 2x_3$$

$$y = \begin{bmatrix} 15 & 7 & 2 \end{bmatrix} \mathbf{x} + 0u$$

b) Verify the realization by recovering the given transfer function from A, B, C, and D.

$$\dot{\mathbf{x}} = A\mathbf{x} + Bu \implies s\mathbf{X}(s) = A\mathbf{X}(s) + BU(s) \implies \mathbf{X}(s) = (sI - A)^{-1}BU(s)$$

$$y = C\mathbf{x} + Du \implies Y(s) = C\mathbf{X}(s) + DU(s) = C(sI - A)^{-1}BU(s) + DU(s)$$

$$\therefore \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D = \begin{bmatrix} 15 & 7 & 2 \end{bmatrix} \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 6 & 11 & s + 6 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 6 & 11 & s + 6 \end{bmatrix}^{-1} = \frac{1}{s(s(s+6)+11)+6} \begin{bmatrix} \dots & \dots & 1 \\ \dots & \dots & s \\ \dots & \dots & s^2 \end{bmatrix} \implies \frac{Y(s)}{U(s)} = \frac{2s^2 + 7s + 15}{s^3 + 6s^2 + 11s + 6}$$

c) Find the eigenvalues of A and classify the zero-input system.

$$\det(\lambda I - A) = 0 \implies \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda + 6 \end{vmatrix} = 0 \implies \lambda(\lambda(\lambda + 6) + 11) + 6 = \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$$

$$(\lambda + 1)(\lambda + 2)(\lambda + 3) = 0 \implies \lambda_1 = -1,\ \lambda_2 = -2 \text{ and } \lambda_3 = -3$$

Zero-input system is a asymptotically stable as all eigenvalues have negative real parts.

Problem 2: State-transition matrix and a step response

Consider

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}.$$

a) Compute the state-transition matrix

$$\Phi(t) = e^{At}$$

analytically and relate its terms to the eigenvalues of A.

$$\Phi(t) = e^{At} = \mathcal{L}^{-1}\{(sI - A)^{-1}\}$$

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 2 & s + 3 \end{bmatrix}^{-1} = \frac{1}{s(s+3)+2} \begin{bmatrix} s + 3 & 1 \\ -2 & s \end{bmatrix}$$

$$\frac{s+3}{s^2+3s+2} = \frac{s+3}{(s+1)(s+2)} = \frac{2}{s+1} + \frac{-1}{s+2}$$

$$\frac{1}{s^2+3s+2} = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} + \frac{-1}{s+2}$$

$$\frac{-2}{s^2+3s+2} = \frac{-2}{(s+1)(s+2)} = \frac{-2}{s+1} + \frac{2}{s+2}$$

$$\frac{s}{s^2+3s+2} = \frac{s}{(s+1)(s+2)} = \frac{-1}{s+1} + \frac{2}{s+2}$$

$$\Phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\det(\lambda I - A) = \lambda(\lambda + 3) + 2 = (\lambda + 1)(\lambda + 2) = 0 \implies \lambda_1 = -1,\ \lambda_2 = -2$$

We get two eigenvalues λ_1, λ_2 and the state-transition matrix has entries that are lienar combinations of $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$.

b) For

$$\mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad u(t) = 1(t),$$

determine $\mathbf{x}(t)$ and $y(t)$. Clearly identify the zero-input and zero-state contributions.

In the following equation, the first part corresponds to the zero-input response and the second part corresponds to the zero-state response:

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}(0) + \int_0^t \Phi(t-\tau)Bu(\tau)\,\mathrm{d}\tau$$

$$\Phi(t)\mathbf{x}(0) = \begin{bmatrix} 2e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\int_0^t \Phi(t-\tau)Bu(\tau)\,\mathrm{d}\tau = \int_0^t \begin{bmatrix} 2e^{-(t-\tau)} - e^{-2(t-\tau)} & e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -2e^{-(t-\tau)} + 2e^{-2(t-\tau)} & -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \mathrm{d}\tau$$

$$= \int_0^t \begin{bmatrix} e^{-(t-\tau)} - e^{-2(t-\tau)} \\ -e^{-(t-\tau)} + 2e^{-2(t-\tau)} \end{bmatrix} \mathrm{d}\tau = \begin{bmatrix} \left[e^{-(t-\tau)} - \frac{1}{2}e^{-2(t-\tau)}\right]_0^t \\ \left[-e^{-(t-\tau)} + e^{-2(t-\tau)}\right]_0^t \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

$$\therefore \mathbf{x}(t) = \begin{bmatrix} \frac{1}{2} + e^{-t} - \frac{1}{2}e^{-2t} \\ -e^{-t} + e^{-2t} \end{bmatrix}$$

c) Find the constant-input equilibrium and verify that it agrees with the limiting value of your response.

$$\dot{\mathbf{x}}_{\text{eq}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = \begin{bmatrix} x_2 \\ -2x_1 - 3x_2 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \mathbf{x}_{\text{eq}} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \qquad \lim_{t \to +\infty} \mathbf{x}(t) = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

d) Explain briefly how the eigenvalues, input, and initial condition appear in the complete response.

e to the power of the eigenvalues appear in the solution as a linear combination. Since both eigenvalues have negative real parts, we have a stable system and we see that our system has an equilibrium point.

Problem 3: The damped nonlinear pendulum

A normalized actuated pendulum is modeled by

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -4\sin(x_1) - x_2 + u,$$

where x_1 is the angular displacement, x_2 is the angular velocity, and u is the applied torque. Take $u_e = 0$.

a) Find all equilibrium points and identify the hanging-down and upright configurations.

$$\dot{\mathbf{x}}_{\text{eq}} = 0 \implies \begin{cases} \dot{x}_{1\text{eq}} = x_{2\text{eq}} = 0 \\ \dot{x}_{2\text{eq}} = -4(\sin x_{1\text{eq}}) - x_{2\text{eq}} + u = 0 \end{cases} \implies \begin{cases} \sin x_{1\text{eq}} = 0 \\ x_{2\text{eq}} = 0 \end{cases} \implies \begin{cases} x_{1\text{eq}} = n\pi, \ n \in \mathbb{Z} \\ x_{2\text{eq}} = 0 \end{cases}$$

If we consider only one rotation, we get 2 soltuions, one where the angular dispalcement is 0 and another where angular dispalcement is π. In both cases angular velocity is zero.

b) Obtain the linearized state-space models about $(0, 0)$ and $(\pi, 0)$.

$$x_1 \approx 0 \text{ and } \sin(x_1) \approx x_1 \implies \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$x_1 \approx \pi \text{ and } \sin(x_1) \approx \pi - x_1 \implies \dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 4 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ -4\pi \end{bmatrix}$$

c) Use the eigenvalues to classify the local behavior at each equilibrium. Support your conclusions with qualitative phase-plane sketches.

$$\begin{vmatrix} \lambda & -1 \\ 4 & \lambda+1 \end{vmatrix} = \lambda(\lambda+1)+4 = \lambda^2+\lambda+4 \implies \lambda_{1,2} = \frac{-1 \pm \sqrt{1-16}}{2}$$

Around the point $x_1 = 0$, we get two distinct eigenvalues with negative real parts, therefore it is a stable equilibrium.

$$\begin{vmatrix} \lambda & -1 \\ -4 & \lambda+1 \end{vmatrix} = \lambda(\lambda+1)-4 = \lambda^2+\lambda-4 \implies \lambda_{1,2} = \frac{-1 \pm \sqrt{1+16}}{2}$$

Around the point $x_1 = \pi$, we get two eigenvalues, one of which is positive, which means it is an unstable equilibrium.

d) Explain physically why the two operating points have different stability properties, and state the limitation of conclusions obtained from linearization.

The first equilibrium point is when the pendulum is hanging, which is stable equilibrium, as a small disturbance in the angular displacement will push the pendulum back to the equilibrium. The second point is the upright configuration, which is a unstable equilibrium, as a change of angular dispalcement will cause the pendulum to move further away from that equilibrium point.

Linearization only allows us to make conclusions around the point of linearization and we cannot get information for the other points.

Problem 4: Eigenvalues, Routh-Hurwitz, and internal stability

Consider the state-space system

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 7 & 1 \\ -3 & 4 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \qquad y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \mathbf{x}.$$

a) Determine the characteristic polynomial and use a Routh table to count the eigenvalues in the left half-plane, right half-plane, and on the imaginary axis.

$$\begin{vmatrix} \lambda-2 & -1 & -1 \\ -1 & \lambda-7 & -1 \\ 3 & -4 & \lambda+5 \end{vmatrix} = (\lambda-2)((\lambda-7)(\lambda+5)-4)-(\lambda+5)+3-4+3(\lambda-7) = \lambda^3-4\lambda^2-33\lambda+51$$

$$\begin{array}{c|cc} \lambda^3 & 1 & -31 \\ \lambda^2 & -4 & 51 \\ \hline \lambda^1 & \frac{1}{4}(51-33\cdot 4) & 0 \\ \lambda^0 & c_1 & 0 \end{array} \implies \begin{array}{c|cc} \lambda^3 & 1 & -31 \\ \lambda^2 & -4 & 51 \\ \hline \lambda^1 & -\frac{81}{4} & 0 \\ \lambda^0 & 51 & 0 \end{array}$$

Two sign changes, which indicate two right half plane poles, no imaginary axis poles, which leaves 1 left half plane pole.